

POLYTOPE NUMBERS AND THEIR PROPERTIES

ABSTRACT. Polytope numbers for a polytope are a sequence of nonnegative integers that are defined by the facial information of a polytope. Every polygon is triangulable and a higher dimensional analogue of this fact states that every polytope is triangulable, namely, every polytope can be decomposed into simplexes. Thus it may be possible to represent polytope numbers by sums of simplex numbers. We analyze a special type of triangulation, called pointed triangulation, and develop several methods to represent polytope numbers by sums of simplex numbers.

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INTRODUCTION

Polygonal numbers are a sequence of nonnegative integers constructed geometrically by a polygon. The square numbers are the numbers of points in square arrays as in Figure 1.

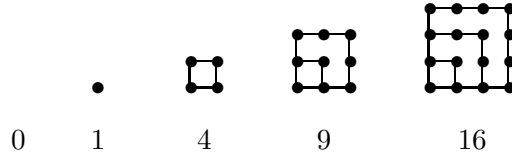


FIGURE 1. Square numbers

Polytope numbers are higher dimensional analogues of polygonal numbers or, equivalently, polygonal numbers are two dimensional polytope numbers. Every polygon is triangulable and a higher dimensional analogue of this fact is that every polytope is triangulable. Thus it may be possible to represent polytope numbers by sums of simplex numbers. We analyze a special type of triangulation, called pointed triangulation, and develop several methods to represent polytope numbers by sums of simplex numbers, which we formulate as decomposition theorems. We also consider several applications of polytope numbers to other mathematical topics.

We provide basic definitions and notations in Section 1. We define pointed triangulation and consider shellings for pointed triangulations in Section 2. We define polytope numbers and provide two descriptions of polytope numbers in Section 3. We furnish several ways of decomposing polytope numbers into simplex numbers and formularize them as decomposition theorems in Section 4. We illustrate decomposition theorems by applying them to several polytopes in Section 5. We suggest applications of polytope numbers in Section 6.

1. PRELIMINARIES

Polytope numbers for a polytope, which we study in Section 3, are a sequence of natural numbers defined by a recurrence relation that uses the facial information of a polytope. Therefore properties of polytope numbers and those of polytopes are closely related. We collect basic material for polytopes that we use in our discussion. We begin with the definition of polytope and provide several examples of polytopes. We next introduce the faces of a polytope, the interior of a polytope, lines in general position, linear functions in general position, polytopal complexes, and shellings of polytopal complexes. The basis of this section is the contents in Ziegler's book [1].

1.1. The definition of polytope. Let \mathbb{R}^d be the vector space of all column vectors of length d with real entries and $(\mathbb{R}^d)^*$ be its dual vector space. The column vectors in \mathbb{R}^d represent points. The column vectors $\mathbf{0}$ and $\mathbf{1}$ are the column vectors of all zeros and all ones, respectively, and the column vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ are the unit vectors in \mathbb{R}^d . Each row vector $\mathbf{a} \in (\mathbb{R}^d)^*$ represents the linear form $l_{\mathbf{a}} : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $l_{\mathbf{a}}\mathbf{x} \mapsto \mathbf{a}\mathbf{x}$. The row vectors $\mathbf{0}$ and $\mathbf{1}$ denote the all zeros and all ones row vectors in $(\mathbb{R}^d)^*$, respectively, and the row vectors $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_d$ are the unit vectors in $(\mathbb{R}^d)^*$. The vector \mathbf{x}^* is the transpose of \mathbf{x} .

The nonempty *affine subspaces* are the translates of linear subspaces. The set of all *affine combinations* of a finite set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is

$$\text{aff}(\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}) = \left\{ \mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} = \sum_{i \in [n]} \lambda_i \mathbf{x}_i \text{ for } \lambda_i \in \mathbb{R}, \sum_{i \in [n]} \lambda_i = 1 \right\},$$

where $[n] = \{1, 2, \dots, n\}$. A set of n points is *affinely independent* if its affine hull has dimension $n - 1$.

For a subset K of \mathbb{R}^d , let $\text{conv}(K)$ be the convex hull of K . A \mathcal{V} -polytope is the convex hull of a finite set of points in some \mathbb{R}^d . An \mathcal{H} -polytope is a bounded intersection of finitely many closed halfspaces in some \mathbb{R}^d . Denoting by $A = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m)^*$ an $m \times d$ -matrix with the rows $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ and writing $\mathbf{z} = (z_1, z_2, \dots, z_m)^*$, we can represent an \mathcal{H} -polytope by

$$P(A, \mathbf{z}) = \{ \mathbf{x} \in \mathbb{R}^d \mid (A, \mathbf{z}) \in \mathbb{R}^{m \times d} \times \mathbb{R}^m, A\mathbf{x} \leq \mathbf{z} \}$$

where the inequality $A\mathbf{x} \leq \mathbf{z}$ is the shorthand for a system of inequalities

$$\mathbf{a}_1 \mathbf{x} \leq z_1, \mathbf{a}_2 \mathbf{x} \leq z_2, \dots, \mathbf{a}_m \mathbf{x} \leq z_m.$$

A *polytope* is a point set that is either a \mathcal{V} -polytope or an \mathcal{H} -polytope.

Theorem 1.1 (Main theorem for polytope [1]). *A subset of \mathbb{R}^d is a \mathcal{V} -polytope if and only if it is an \mathcal{H} -polytope.*

The dimension of a polytope P , denoted by $\dim(P)$, is the dimension of its affine hull and a d -polytope is a polytope of dimension d in some \mathbb{R}^e with $e \geq d$.

Some recycling operations produce new polytopes. Let P be a d -polytope and \mathbf{x}_0 be a point outside of $\text{aff}(P)$ (for this we embed P into \mathbb{R}^n for some $n > d$). A *pyramid* over P is

$$\text{pyr}(P) = \text{conv}(P \cup \{\mathbf{x}_0\}).$$

The face set of $\text{pyr}(P)$ is

$$\{F, \text{pyr}(F) \mid F \text{ is a face of } P\}.$$

Similarly, a *bipyramid* over P is

$$\text{bipyr}(P) = \text{conv}(P \cup \{\mathbf{x}_-, \mathbf{x}_+\})$$

where both \mathbf{x}_- and \mathbf{x}_+ are in outside of $\text{aff}(P)$ and an interior point of the segment $\text{conv}(\{\mathbf{x}_-, \mathbf{x}_+\})$ is an interior point of P .

For two polytopes P and P' the *product* of P and P' is

$$P \times P' = \{(\mathbf{x}, \mathbf{x}')^* \mid \mathbf{x} \in P, \mathbf{x}' \in P'\}.$$

The dimension of $P \times P'$ is $\dim(P) + \dim(P')$ and the face set of $P \times P'$ is

$$\{F \times F' \mid F \text{ (resp. } F') \text{ is a face of } P \text{ (resp. } P').\}$$

Example 1.1. A d -simplex α^d is the convex hull of $d+1$ affinely independent points in \mathbb{R}^n with $n \geq d$. Thus a d -simplex is a polytope of dimension d with $d+1$ vertexes and it is a pyramid over a $(d-1)$ -simplex. The *standard d -simplex* α_s^d is the simplex in \mathbb{R}^{d+1} defined by

$$\alpha_s^d = \text{conv}(\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{d+1}\}).$$

We construct a d -cross polytope β^d by an iteration. Let β^0 be a point. For $d \geq 1$ we define a d -cross polytope to be $\beta^d = \text{bipyrr}(\beta^{d-1})$. The *standard d -cross polytope* β_s^d is the cross polytope defined by

$$\beta_s^d = \text{conv}(\{\pm \mathbf{e}_1, \pm \mathbf{e}_2, \dots, \pm \mathbf{e}_d\}).$$

We also form a d -measure polytope γ^d by an iteration. Let γ^0 be a point and γ^1 be a line segment. For $d \geq 2$ we define $\gamma^d = \gamma^{d-1} \times \gamma^1$. In particular, the *standard d -measure polytope* γ_s^d is the measure polytope defined by

$$\gamma_s^d = \text{conv}\left(\left\{\sum_{i \in [d]} a_i \mathbf{e}_i \mid a_i \in \{1, -1\}\right\}\right).$$

1.2. The faces and the interior of a polytope. For a polytope P we define $\text{vert}(P)$ to be the vertex set of P and $\mathcal{F}(P)$ (resp. $\mathcal{F}_k(P)$) to be the face (resp. k -face) set of P .

For $\mathbf{y} \in P$ if every proper face of P does not contain \mathbf{y} , then we say that \mathbf{y} is an *interior point* of P . We define $\text{int}(P)$ to be the *interior* of P , which is the set of all interior points in P , and $\partial P = P \setminus \text{int}(P)$ to be the *boundary* of P . We call $\text{relint}(P)$ the *relative interior* of P , which is the interior of P with respect to an embedding of P into its affine hull where P is full dimensional. Analogous to the interior of a polytope, $\text{relint}(P)$ is the set

of points in P that are in no proper face of P . By the definition of relative interior,

$$P = \bigsqcup_{F \in \mathcal{F}(P)} \text{relint}(F)$$

where \bigsqcup denotes the disjoint union.

1.3. Lines and linear functions in general position. Let $P = P(A, \mathbf{1})$ be a d -polytope with $A = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)^*$. A line through $\mathbf{0} \in \text{int}(P)$ is in *general position* with respect to P if it is not parallel to any hyperplane that defines faces of P and it does not hit the intersection of any two of them. If we write the line in the form $L(\mathbf{u}) = \{t\mathbf{u} \mid t \in \mathbb{R}\}$ for some $\mathbf{u} \neq \mathbf{0}$, then general position means that $\mathbf{a}_i\mathbf{u} \neq \mathbf{a}_j\mathbf{u}$ when $i \neq j$ and $\{i, j\} \subseteq [n]$.

Lemma 1.2 (Ziegler [1]). *Let $P = P(A, \mathbf{1})$ and $\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. If λ is small enough, then the line $L(\mathbf{u}^\lambda)$ is in general position with respect to P where*

$$\mathbf{u}^\lambda = \mathbf{u} + (\lambda, \lambda^2, \dots, \lambda^d)^*.$$

Corollary 1.3 (Ziegler [1]). *For a polytope P a line in general position with respect to P exists.*

A linear function $\mathbf{c}\mathbf{x}$ is in general position with respect to a polytope P if it separates the vertexes of P , that is, if $\mathbf{c}\mathbf{v}_i \neq \mathbf{c}\mathbf{v}_j$ for any two distinct vertexes \mathbf{v}_i and \mathbf{v}_j of P .

Lemma 1.4 (Ziegler [1]). *Let $P = P(A, \mathbf{1})$ and $\mathbf{c} \in (\mathbb{R}^d)^* \setminus \{\mathbf{0}\}$. If $\lambda > 0$ is small enough, then the linear function $\mathbf{c}^\lambda\mathbf{x}$ is in general position with respect to P where*

$$\mathbf{c}^\lambda = \mathbf{c} + (\lambda, \lambda^2, \dots, \lambda^d).$$

Corollary 1.5 (Ziegler [1]). *For a polytope P a linear function in general position with respect to P exists.*

Let P be a polytope with $\mathbf{0} \in \text{int}(P)$ and F be a proper face of P . For a point $\mathbf{y} \in \mathbb{R}^d$ we say that \mathbf{y} is a point *beyond* F if \mathbf{y} and $\mathbf{0}$ lie on different sides of H_1 for every facet defining hyperplane H_1 that includes F , but on the same side of H_2 for every facet defining hyperplane H_2 that does not include F .

Let P be a polytope. The vertexes and edges of P form a finite, undirected, and simple graph $G(P)$, called the *graph* of P . For every face F of P we define $G(F)$ to be the induced subgraph of $G(P)$ on $\text{vert}(F)$, that is, the graph of all vertexes in F and all edges of P between them. The graph $G(F)$ coincides with the graph of F if we consider F as a polytope.

We consider an orientation of $G(P)$ that assigns a direction to every edge. An orientation is *acyclic* if no directed cycle is in it. Thus, if a graph $G(P)$ has an acyclic orientation, then $G(P)$ has a *sink*, that is, a vertex that does not have an edge directed away from it. If a linear function \mathbf{cx} is in general position, then this linear function gives a well-defined method to direct the graph $G(P)$, by directing an edge $\text{conv}(\{\mathbf{v}_i, \mathbf{v}_j\})$ from \mathbf{v}_i to \mathbf{v}_j if $\mathbf{cv}_i > \mathbf{cv}_j$. We call this orientation the *orientation* of $G(P)$ induced by \mathbf{cx} . Monotone paths on P , that is, edge paths such that the objective function increases strictly in each step, translate into directed paths in the orientation of $G(P)$ induced by \mathbf{cx} .

Theorem 1.6 (Ziegler [1]). *For a polytope P , if \mathbf{cx} is a linear function in general position for P , then the orientation of $G(P)$ induced by \mathbf{cx} is acyclic with a unique sink. This sink is the unique point in P such that \mathbf{cx} achieves its minimum.*

Note that if O is an acyclic orientation of $G(P)$, then the restriction of $G(P)$ to each nonempty subset A of $\text{vert}(P)$ has a sink with respect to O .

An acyclic orientation O of $G(P)$ is *good* if for every nonempty face F of P the graph $G(F)$ has exactly one sink. The existence of good acyclic orientations of $G(P)$ follows from Theorem 1.6.

Corollary 1.7 (Ziegler [1]). *For a polytope P , if \mathbf{cx} is in general position for P , then it is in general position for each face of P .*

1.4. Polytopal complexes and shellings. A *polytopal complex* \mathcal{C} is a finite collection of polytopes that satisfies the following conditions:

Condition 1. The empty polytope is in \mathcal{C} .

Condition 2. If P is an element of \mathcal{C} , then every face of P is also in \mathcal{C} .

Condition 3. The intersection of two polytopes P and P' in \mathcal{C} is a face both of the polytopes P and P' .

The dimension of \mathcal{C} is defined by

$$\dim(\mathcal{C}) = \max\{\dim(P) \mid P \in \mathcal{C}\}$$

and the *underlying set* of \mathcal{C} is the point set $|\mathcal{C}| = \bigcup_{P \in \mathcal{C}} P$.

A polytopal complex is *pure* if each of its faces is in a face of dimension $\dim(\mathcal{C})$, that is, if all of the inclusion maximal faces of \mathcal{C} , called the *facets* of \mathcal{C} , have the same dimension. Let \mathcal{C}_P be the complex formed by the faces of a polytope P . The *boundary complex* $\mathcal{C}_{\partial P}$ is the subcomplex of \mathcal{C}_P formed by all proper faces of \mathcal{C}_P . Thus its underlying set is

$$|\mathcal{C}_{\partial P}| = \partial P = P \setminus \text{relint}(P).$$

A *subdivision* of a polytope P is a polytopal complex \mathcal{C}_P with the underlying space $|\mathcal{C}_P| = P$. The subdivision is a *triangulation* if every polytope in \mathcal{C}_P is a simplex.

Although a polytopal complex is a set of polytopes, we use it as a generalization of a polytope and we define the faces of a polytopal complex to be its elements. For example, the faces of a hexagon's triangulation in Figure 2 are the vertexes, the edges, and the triangles in this triangulation.

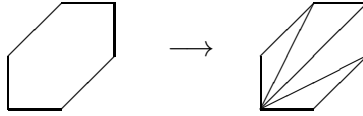


FIGURE 2. A hexagon's triangulation

From this generalization, we denote

$$\mathcal{C} = \bigoplus_{F \in \mathcal{C}} F$$

and define $\mathcal{F}_k(\mathcal{C})$ to be the set of k -faces of \mathcal{C} . Since every face of a polytopal complex \mathcal{C} is in a facet of \mathcal{C} , if $\dim(\mathcal{C}) = d$ and \mathcal{C} is a pure polytopal complex,

then we denote

$$\mathcal{C} = \bigoplus_{F \in \mathcal{F}_d(\mathcal{C})} F.$$

Note that

$$\mathcal{C} = \bigoplus_{F \in \mathcal{C}} F \text{ and } \mathcal{C} = \bigoplus_{F \in \mathcal{F}_d(\mathcal{C})} F$$

are polytopal complex analogues of

$$|\mathcal{C}| = \bigcup_{F \in \mathcal{C}} F \text{ and } |\mathcal{C}| = \bigcup_{F \in \mathcal{F}_d(\mathcal{C})} F,$$

respectively.

Let \mathcal{C}_P be a polytopal complex obeying $|\mathcal{C}_P| = P$. For $F \in \mathcal{F}(P)$ the subdivision \mathcal{C}_F of F on \mathcal{C}_P is

$$\mathcal{C}_F = \{F' \in \mathcal{C}_P \mid F' \subseteq F\}.$$

Therefore

$$\mathcal{C}_F = \bigoplus_{\substack{F' \in \mathcal{C}_P \\ F' \subseteq F}} F'.$$

We call \mathcal{C}_F the subdivision of F with respect to \mathcal{C}_P .

For a pure d -polytopal complex \mathcal{C} a *shelling* of \mathcal{C} is a linear ordering of the facets F_1, F_2, \dots, F_s in \mathcal{C} such that either \mathcal{C} is 0-dimensional or it satisfies the following conditions:

Condition 1. The boundary complex $\mathcal{C}_{\partial F_1}$ of the first facet F_1 has a shelling.

Condition 2. For $j \in [s] \setminus \{1\}$ the intersection of the facet F_j with the union of previous facets is a nonempty set and a beginning segment of a shelling of the $(d-1)$ -dimensional boundary complex in F_j , that is,

$$F_j \cap \left(\bigcup_{i \in [j-1]} F_i \right) = F'_1 \cup F'_2 \cup \dots \cup F'_r$$

for some shelling $F'_1, F'_2, \dots, F'_r, \dots, F'_{s'}$ of $\mathcal{C}_{\partial F_j}$. In particular, this condition requires that $F_j \cap \left(\bigcup_{i \in [j-1]} F_i \right)$ has a shelling, therefore it has to be pure $(d-1)$ -dimensional and connected for $j \in [s] \setminus \{1\}$.

A polytopal complex is *shellable* if it is pure and has a shelling.

Remark 1.1.

1. Every simplex is shellable and every ordering of its facets is a shelling.

These facts immediately follow by induction on dimension since the intersection of F_j with F_i for $i < j$ is always a facet of F_j .

2. For triangulations, the shelling condition 1 is redundant. Thus we can simplify the shelling condition 2 considerably. We can replace the shelling condition 2 with

Condition 2'. For $j \in [s] \setminus \{1\}$ the intersection of the facet F_j with the previous facets is nonempty and pure $(d-1)$ -dimensional. In other words, for every $i < j$ there exists some $l < j$ such that $F_i \cap F_j$ is a subpolytope of $F_l \cap F_j$ and $F_l \cap F_j$ is a facet of F_j .

Let P be a polytope and \mathbf{x} be a point. The point \mathbf{x} lies in *general position* with respect to the polytope P if \mathbf{x} is not in the affine hull of a facet in P . A facet F of a P is *visible* from \mathbf{x} if for every $\mathbf{y} \in F$ the line segment $\text{conv}(\{\mathbf{x}, \mathbf{y}\})$ intersects P only in the point \mathbf{y} . Equivalently, F is visible from \mathbf{x} if and only if \mathbf{x} and $\text{int}(P)$ are on different sides of the hyperplane $\text{aff}(F)$ spanned by F . For example, if \mathbf{x}_G is beyond the face G , then the facets that include G are exactly those that are visible from \mathbf{x}_G .

Theorem 1.8 (Bruggersser [2]). *Let P be a polytope and \mathbf{x} be a point outside P . If point \mathbf{x} lies in general position with respect to P , then the boundary complex $\mathcal{C}_{\partial P}$ of P has a shelling such that the facets of P that are visible from \mathbf{x} come first.*

This theorem shows that every polytope is shellable. In Section 2, we need to use Ziegler's proof of Theorem 1.8 [1], thus we provide that proof here.

Proof. Let \mathbf{x} be a point that lies in general position with respect to P . We choose a line l through both \mathbf{x} and a point in general position for P . The properties we need are that l contains \mathbf{x} , l hits the interior of P , and any two different facet-defining hyperplanes H and H' of P satisfy $l \cap H \neq l \cap H'$. For simplicity, we assume that l is not parallel to any of the facet hyperplanes, thus there is no intersection point at infinity. We orient l from P to \mathbf{x} .

Now we imagine that P is a polyhedral planet and there is a rocket that starts on its surface at the point where the oriented line l leaves the planet. This point lies on a unique facet F_1 of P and for the first few minutes of the flight only F_1 is visible from the rocket.

After a while, a new facet appears on the horizon. The rocket passes through a hyperplane H_2 and we label the corresponding facet F_2 . We continue to label the facets F_3, F_4, \dots of P in the order where the rocket passes through their hyperplanes, that is, in the order such that the facets appear on the horizon, becoming visible from the rocket. Now we imagine that the rocket passes through and comes back to the planet from the opposite side. We continue the shelling by taking the facets in the order such that the rocket passes through the hyperplanes $\text{aff}(F_i)$, that is, the corresponding facets disappear on the horizon.

This rocket flight clearly gives a well-defined ordering on the facets of P . What's more, the facets that are visible from \mathbf{x} form a beginning segment, since we see exactly those facets at the point where the rocket passes through \mathbf{x} .

To verify that the ordering is a shelling, we consider the intersection

$$\partial F_j \cap \left(\bigcup_{i \in [j-1]} F_i \right).$$

If we add F_j before we pass through infinity, then this intersection is exactly the set of those facets in F_j that are visible from the point $l \cap \text{aff}(F_j)$ where F_j appears on the horizon. Thus we know by induction on dimension that this collection of facets of F_j is shellable and can be continued to a shelling of the whole boundary ∂F_j .

After the rocket passes through infinity, the intersection is the family of nonvisible facets. This family of nonvisible facets is also shellable because reversing the orientation of l yields the shelling with the reversed ordering of the facets. \square

Corollary 1.9 (Ziegler [1]). *For any two facets F and F' of a polytope P , there is a shelling of ∂P such that F is the first facet and F' is the last one.*

Proof. We choose a shelling line l that intersects the boundary of P in the facets F and F' . For example, we choose two points \mathbf{x} and \mathbf{x}' beyond F and F' , respectively, and let l be the line determined by \mathbf{x} and \mathbf{x}' . If necessary, we perturb l to general position. \square

2. POINTED TRIANGULATIONS

Every polygon is triangulable and we may regard a polytope's triangulation as a higher dimensional analogue of a polygon's triangulation. We introduce a special kind of triangulation, called pointed triangulation, and study shellings of this triangulation. Pointed triangulations and their shellings constitute the main tool to relate polytopes and polytope numbers.

Let \mathcal{C}_P be a triangulation of a d -polytope P that satisfies the following conditions: For $d \in \mathbb{N} = \{0, 1, \dots\}$ let $[d]_0 = \{0, 1, \dots, d\}$.

Condition 1. For $k \in [d]_0$ each k -face F of P has a triangulation

$$\mathcal{C}_F = \bigoplus_{\alpha^k \in \mathcal{C}_F} \alpha^k$$

such that there is a designated vertex $\mathbf{v}_F \in \text{vert}(F)$, called the *apex* of F , satisfying

$$\mathbf{v}_F \in \bigcap_{\alpha^k \in \mathcal{C}_F} \alpha^k.$$

Condition 2. For any two faces F_1 and F_2 of P if $\{\mathbf{v}_{F_1}, \mathbf{v}_{F_2}\} \subseteq F_1 \cap F_2$ then

$$\mathbf{v}_{F_1} = \mathbf{v}_{F_2}.$$

Condition 3. For each face F of P if $\mathbf{w} \in \text{vert}(F) \setminus \{\mathbf{v}_F\}$ then the edge $\text{conv}(\{\mathbf{v}_F, \mathbf{w}\})$ is in \mathcal{C}_F .

We define

$$V(P) = \{\mathbf{v}_F \mid F \in \mathcal{F}(P)\}$$

and call it the *set of apexes* of \mathcal{C}_P . Note that $V(P)$ is a multiset by the condition 2. Even though the set $V(P)$ is dependent on a pointed triangulation of P , we use $V(P)$ by abusing notation. We call \mathcal{C}_P the $V(P)$ -*pointed triangulation*. Conditions 1–3 are called the *pointed triangulation conditions*.

Theorem 2.1. *Every polytope has a pointed triangulation.*

Proof. Let P be a d -polytope. By Lemma 1.5 a linear function in general position $\mathbf{c}\mathbf{x}$ with respect to P exists and by Corollary 1.7 the linear function $\mathbf{c}\mathbf{x}$ decides a unique sink \mathbf{v}_F for each face $F \in \mathcal{F}(P)$. For each $F \in \mathcal{F}(P)$ we define

$$V(F) = \{\mathbf{v}_G \mid G \in \mathcal{F}(F)\}$$

and

$$\mathcal{C}_F = \{\emptyset\} \cup \left\{ \text{conv}(\{\mathbf{v}_{G_1}, \mathbf{v}_{G_2}, \dots, \mathbf{v}_{G_k}\}) \mid k \in [d], \right. \\ \left. G_i \in \mathcal{F}(F), G_i \supsetneq G_{i+1}, \mathbf{v}_{G_i} \notin G_{i+1} \right\}.$$

We claim that \mathcal{C}_P is the $V(P)$ -pointed triangulation. To verify this claim, we use induction on dimension.

We first show that \mathcal{C}_P is a triangulation of P . By the definition of \mathcal{C}_P the polytopal complex \mathcal{C}_P is a triangulation, thus we need only show that $|\mathcal{C}_P| = P$. Moreover, $|\mathcal{C}_P| \subseteq P$, therefore it suffices to show that $|\mathcal{C}_P| \supseteq P$.

Let $\mathbf{x} \in P$. Since

$$P = \bigcup_{\substack{F \in \mathcal{F}_{d-1}(P) \\ \mathbf{v}_P \notin F}} \text{conv}(\{\mathbf{v}_P\} \cup F),$$

there is a facet F of P such that

$$\begin{cases} \mathbf{v}_P \notin F \\ \mathbf{x} \in \text{conv}(\{\mathbf{v}_P\} \cup F) \end{cases}.$$

By the way, for each $F \in \mathcal{C}$ the complex \mathcal{C}_F satisfies $|\mathcal{C}_F| = F$ by the induction hypothesis, hence there is a $(d-1)$ -simplex α^{d-1} in \mathcal{C}_F such that

$$\mathbf{x} \in \text{conv}(\{\mathbf{v}_P\} \cup \alpha^{d-1}).$$

By the definition of \mathcal{C}_P the polytope $\text{conv}(\{\mathbf{v}_P\} \cup \alpha^{d-1})$ is an element of \mathcal{C}_P , thus $\mathbf{x} \in |\mathcal{C}_P|$. It follows that $|\mathcal{C}_P| \supseteq P$.

We now show that \mathcal{C}_P is the $V(P)$ -pointed triangulation. By the definition of \mathcal{C}_P , the complex \mathcal{C}_P satisfies the pointed triangulation conditions 1 and 3. Let F_1 and F_2 be two faces of P such that $\{\mathbf{v}_{F_1}, \mathbf{v}_{F_2}\} \subseteq F_1 \cap F_2$. Since

$$\mathbf{v}_{F_1} \in F_1 \cap F_2 \in \mathcal{F}(F_1),$$

we obtain $\mathbf{v}_{F_1 \cap F_2} = \mathbf{v}_{F_1}$, and similarly, $\mathbf{v}_{F_1 \cap F_2} = \mathbf{v}_{F_2}$. This yields $\mathbf{v}_{F_1} = \mathbf{v}_{F_2}$. As a result, \mathcal{C}_P satisfies the pointed triangulation condition 2. This proves that \mathcal{C}_P is the $V(P)$ -pointed triangulation. \square

From now on, every pointed triangulation is formed by the method in the proof of Theorem 2.1.

Theorem 2.2. *Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P . There is a shelling of \mathcal{C}_P such that for $\mathcal{F}_d(\mathcal{C}_P) = \{\alpha_1^d, \alpha_2^d, \dots, \alpha_s^d\}$ if $j \in [s] \setminus \{1\}$, then the number of facets of α_j^d in $\alpha_j^d \cap \left(\bigcup_{i \in [j-1]} \alpha_i^d \right)$, denoted by l_j , satisfies $l_j \in [q-1]$.*

Proof. The complex \mathcal{C}_P is a triangulation, thus we need only show the shelling condition 2'. We borrow the definitions and notations in the proof of Theorem 1.8.

Let

$$\mathcal{F}_{d-1}(\mathcal{C}_P, \widehat{\mathbf{v}_P}) = \{\alpha_1^{d-1}, \alpha_2^{d-1}, \dots, \alpha_s^{d-1}\}$$

be the set of $(d-1)$ -simplexes α^{d-1} on $\partial(P)$ satisfying

$$\begin{cases} \mathbf{v}_P \notin \alpha^{d-1} \\ \alpha_i^d = \text{conv}(\{\mathbf{v}_P\} \cup \alpha_i^{d-1}) \end{cases}.$$

Since \mathcal{C}_P is a pointed triangulation, we may instead prove that there is a shelling of $\mathcal{F}_{d-1}(\mathcal{C}_P, \widehat{\mathbf{v}_P})$ such that the number of facets of α_j^{d-1} in $\alpha_j^{d-1} \cap \left(\bigcup_{i \in [j-1]} \alpha_i^{d-1} \right)$, denoted by l_j , satisfies $l_j \in [d-1]$. We define such a shelling to be the *triangulation shelling*.

Without loss of generality, we assume that $\mathbf{f}_d \mathbf{x}$ is a linear function in general position that determines $V(P)$. We choose a point \mathbf{x} that lies in general position, that is, does not lie in the affine hull of a facet of P , and a line l through \mathbf{x} and a point in general position. By Corollary 1.9, we may assume that l passes through two distinct facets F^{d-1} and F_P^{d-1} of P such that $\mathbf{v}_m \in F^{d-1}$ and $\mathbf{v}_P \in F_P^{d-1}$ where

$$\begin{cases} \mathbf{f}_d \mathbf{v}_m = \max\{\mathbf{f}_d \mathbf{v} \mid \mathbf{v} \in \text{vert}(P)\} \\ \mathbf{f}_d \mathbf{v}_P = \min\{\mathbf{f}_d \mathbf{v} \mid \mathbf{v} \in \text{vert}(P)\} \end{cases}.$$

We may further assume by Lemma 1.2 that the line l is orthogonal to the plane $\mathbf{f}_d \mathbf{x} = 0$ and the rocket defined by l moves in the direction that the value of d th coordinate in its position vector increases. As in Theorem 1.8, this rocket flight assigns a shelling to the facets of P , thus it assigns a shelling to those facets that do not contain \mathbf{v}_P .

If either $d = 0$ or $d = 1$, then the line l evidently assigns a triangulation shelling to $\mathcal{F}_{d-1}(\mathcal{C}_P, \widehat{\mathbf{v}_P})$. Suppose that $d \geq 2$. We assume that the line l assigns a triangulation shelling to every pointed triangulation of a k -polytope when $k \in [d-1] \cup \{0\}$.

We first assign a shelling to the facets of P that do not contain \mathbf{v}_P as in Theorem 1.8. Let them be $F_1^{d-1}, F_2^{d-1}, \dots$ and $\mathcal{C}_{F_i^{d-1}}$ be the $V(F_i^{d-1})$ -pointed triangulation. In this case, the indexes in the facets of P equal those in the shelling of P . By the induction hypothesis, $\mathcal{C}_{F_1^{d-1}}$ has a triangulation shelling. Suppose that $\bigcup_{i \in [k]} \mathcal{C}_{F_i^{d-1}}$ has a triangulation shelling and let $\mathbf{v}_{F_{k+1}^{d-1}}$ be the sink of F_{k+1}^{d-1} . Then

$$\mathbf{v}_{F_{k+1}^{d-1}} \notin \bigcup_{i \in [k]} F_i^{d-1} \text{ or } \mathbf{v}_{F_{k+1}^{d-1}} \in \bigcup_{i \in [k]} F_i^{d-1}.$$

We suppose that $\mathbf{v}_{F_{k+1}^{d-1}} \notin \bigcup_{i \in [k]} F_i^{d-1}$ and define

$$C_{F_{k+1}^{d-1}} = \bigoplus_{i \in [m_{k+1}]} \alpha_{k+1,i}^{d-1}.$$

If we use induction on dimension by considering the formation of $\bigcup_{i \in [k]} \mathcal{C}_{F_i^{d-1}}$, then the triangulation $\mathcal{C}_{F_{k+1}^{d-1}}$ has a shelling such that the number of facets of $\alpha_{k+1,i}^{d-1}$ in

$$\left| \bigcup_{i \in [k]} \mathcal{C}_{F_i^{d-1}} \right| \bigcup \left(\bigcup_{j \in [i]} \alpha_{k+1,j}^{d-1} \right)$$

is at most $d-1$.

Suppose that $\mathbf{v}_{F_{k+1}^{d-1}} \in \bigcup_{i \in [k]} F_i^{d-1}$. If every facet $F_{k+1,1}^{d-2}$ of F_{k+1}^{d-1} obeying

$$\dim \left(\left| \bigcup_{i \in [k]} \mathcal{C}_{F_i^{d-1}} \right| \cap F_{k+1,1}^{d-2} \right) = d-2$$

contains the apex $\mathbf{v}_{F_{k+1}^{d-1}}$, then $\bigcup_{i \in [k+1]} \mathcal{C}_{F_i^{d-1}}$ has a triangulation shelling. Assume that F_{k+1}^{d-1} has another facet $F_{k+1,2}^{d-2}$ satisfying

$$\begin{cases} F_{k+1,2}^{d-2} \subseteq \bigcup_{i \in [k]} F_i^{d-1} \\ \dim(F_{k+1,1}^{d-2} \cap F_{k+1,2}^{d-2}) = d-3 \\ \mathbf{v}_{F_{k+1}^{d-1}} \notin F_{k+1,2}^{d-2} \end{cases} \quad .$$

Using Corollary 1.9, we can choose a shelling of $\partial(F_{k+1}^{d-1})$ such that $F_{k+1,1}^{d-2}$ is the last facet and $F_{k+1,2}^{d-2}$ is the first one. If we use induction on dimension by considering the formation of $\bigcup_{i \in [k]} \mathcal{C}_{F_i^{d-1}}$, then we can assign a triangulation shelling to $\bigcup_{i \in [k+1]} \mathcal{C}_{F_i^{d-1}}$.

Considering all of the facets in $\mathcal{F}_{d-1}(\mathcal{C}_P, \widehat{\mathbf{v}_P})$ allows us to assign a triangulation shelling to these facets. \square

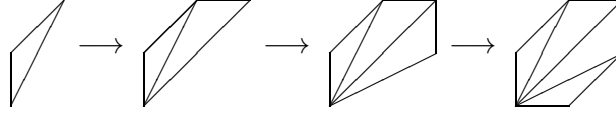


FIGURE 3. A triangulation shelling of a pointed triangulation

3. POLYTOPE NUMBERS

We define polytope numbers in this section. By using the definition of polytope numbers, we also derive the product formula for polytope numbers. Intuitively, polytope numbers for a polytope are a sequence of numbers associated with a polytope. To count these numbers more effectively, we need a canonical method to describe them geometrically, called the geometric description of polytope numbers. We begin with the geometric description of simplex numbers, introduce the facet-cut, suggest the geometric description of polytope numbers, and finally consider the description of polytope numbers by vertex sets.

3.1. The definition of polytope numbers. For a d -polytope we define a sequence of *polytope numbers* $P(n)$ and *interior polytope numbers* $P(n)^\sharp$ by double induction on d and n .

When $d = 0$, we define

$$\begin{cases} P(0) = 0, P(n) = 1 \text{ for } n \geq 1 \\ P(0)^\sharp = 0, P(n)^\sharp = 1 \text{ for } n \geq 1 \end{cases}.$$

When $d \geq 1$, we suppose that for each k -polytope F satisfying $k < d$ the numbers $F(n)$ and $F(n)^\sharp$ are defined. We define

$$\begin{cases} P(0) = 0, P(1) = 1 \\ P(n) = P(n-1) + \sum_{\mathbf{v}_P \notin F \in \mathcal{F}(P)} F(n)^\sharp \text{ for } n \geq 2 \end{cases}$$

and

$$\begin{cases} P(0)^\sharp = 0, P(1)^\sharp = 0 \\ P(n)^\sharp = P(n) - \sum_{F \in \mathcal{F}(P) \setminus \{P\}} F(n)^\sharp \text{ for } n \geq 2 \end{cases},$$

where \mathbf{v}_P is a fixed vertex of P . Following the definition of polytope numbers, we can also define

$$P(n) = \sum_{F \in \mathcal{F}(P)} F(n)^\sharp.$$

Remark 3.1.

1. To define polytope numbers for a polytope P , we need to choose a vertex \mathbf{v}_P of P and a vertex \mathbf{v}_F of F for each $F \in \mathcal{F}(P) \setminus \{P\}$ with $\mathbf{v}_P \notin F$. Defining $V(P)$ to be the set of such vertexes, we call $P(n)$ the $V(P)$ -polytope numbers.
2. The set $V(P)$ for the $V(P)$ -polytope numbers coincides with the set $V(P)$ for the $V(P)$ -pointed triangulation defined in Section 2. The geometric description of polytope numbers explains the reason for this coincidence.

3.2. Polytope numbers for products of polytopes. Following the definition of polytope numbers and that of the product of polytopes, we can expect that polytope and interior polytope numbers for the product of two polytopes are the product of two corresponding polytope and interior polytope numbers, respectively. We shot that this is actually true.

For $i \in [2]$ let P_i be a d_i -polytope. Suppose that $P_i(n)$ are the $V(P_i)$ -polytope numbers and $P_1 \times P_2(n)$ are the $V(P_1 \times P_2)$ -polytope numbers. We claim that

$$(3.1) \quad \begin{cases} P_1 \times P_2(n) = P_1(n) \times P_2(n) \\ P_1 \times P_2(n)^\# = P_1(n)^\# \times P_2(n)^\# \end{cases}.$$

To prove this claim, we use double induction on the numbers $d = d_1 + d_2$ and n .

If either $d = 0$ or $n = 0$, then the identity (3.1) is trivially true. Suppose that $d \geq 1$ and $n \geq 1$. The apex of $P_1 \times P_2$ is $\mathbf{v}_{P_1} \times \mathbf{v}_{P_2}$, thus

$$(3.2) \quad P_1 \times P_2(n) = P_1 \times P_2(n-1) + \sum_{\substack{\mathbf{v}_{P_1} \times \mathbf{v}_{P_2} \notin F_1 \times F_2 \\ F_1 \times F_2 \in \mathcal{F}(P_1 \times P_2)}} F_1 \times F_2(n)^\#$$

By the induction hypothesis

$$\begin{cases} P_1 \times P_2(n-1) = P_1(n-1) \times P_2(n-1) \\ F_1 \times F_2(n)^\# = F_1(n)^\# \times F_2(n)^\# \end{cases},$$

hence the identity (3.2) becomes

$$P_1 \times P_2(n) = P_1(n-1) \times P_2(n-1) + \sum_{\substack{\mathbf{v}_{P_1} \times \mathbf{v}_{P_2} \notin F_1 \times F_2 \\ F_1 \times F_2 \in \mathcal{F}(P_1 \times P_2)}} F_1(n)^\# \times F_2(n)^\#$$

In addition,

$$\begin{aligned} P_1(n-1) \times P_2(n-1) &= \sum_{\substack{\mathbf{v}_{P_1} \in F_1 \\ F_1 \in \mathcal{F}(P_1)}} F_1(n)^\# \times \sum_{\substack{\mathbf{v}_{P_2} \in F_2 \\ F_2 \in \mathcal{F}(P_2)}} F_2(n)^\# \\ &= \sum_{\substack{\mathbf{v}_{P_1} \times \mathbf{v}_{P_2} \in F_1 \times F_2 \\ F_1 \times F_2 \in \mathcal{F}(P_1 \times P_2)}} F_1(n)^\# \times F_2(n)^\#, \end{aligned}$$

therefore

$$\begin{aligned} P_1 \times P_2(n) &= \sum_{F_1 \times F_2 \in \mathcal{F}(P_1 \times P_2)} F_1(n)^\# \times F_2(n)^\# \\ &= \left(\sum_{F_1 \in \mathcal{F}(P_1)} F_1(n)^\# \right) \times \left(\sum_{F_2 \in \mathcal{F}(P_2)} F_2(n)^\# \right) = P_1(n) \times P_2(n). \end{aligned}$$

It follows that

$$P_1 \times P_2(n) = P_1(n) \times P_2(n).$$

Similarly,

$$P_1 \times P_2(n)^\sharp = P_1(n)^\sharp \times P_2(n)^\sharp.$$

Generalizing the identity (3.1) to the product of several polytopes, we can compute polytope numbers for products of several polytopes.

Theorem 3.1 (Polytope numbers for the product of polytopes). *Let P_1, P_2, \dots, P_l be polytopes. Suppose that $P_i(n)$ are the $V(P_i)$ -polytope numbers. If $\prod_{i \in [l]} P_i(n)$ are the $V\left(\prod_{i \in [l]} P_i\right)$ -polytope numbers, then*

$$\begin{cases} \left(\prod_{i \in [l]} P_i\right)(n) = \prod_{i \in [l]} P_i(n) \\ \left(\prod_{i \in [l]} P_i\right)(n)^\sharp = \prod_{i \in [l]} P_i(n)^\sharp \end{cases}.$$

3.3. The geometric description of simplex numbers and the facet-cut. To define polytope numbers for a polytope P , we need to use its facial information. Hence it may be possible to arrange points in P that correspond to polytope numbers for P . Since polytope numbers for P are determined by a pointed triangulation \mathcal{C}_P , we need to consider a method to describe simplex numbers by such arrangements of points in simplexes. We proceed by induction.

We first consider polytope numbers for the standard d -simplex α_s^d . We claim that

$$(3.3) \quad \alpha_s^d(n) = \left| \left\{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n - 1 \right\} \right|.$$

If either $d = 0$ or $(d, n) \in (\mathbb{N} \setminus \{0\}) \times [n]_1$, then the identity (3.3) is obviously true. For $d \geq 2$ and $n \geq 2$, we suppose that

$$\begin{cases} \alpha_s^k(n) = \left| \left\{ \mathbf{x} \in \mathbb{N}^{k+1} \mid \mathbb{1}\mathbf{x} = n - 1 \right\} \right| & \text{when } k \in [d-1] \\ \alpha_s^d(m) = \left| \left\{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = m - 1 \right\} \right| & \text{when } m \in [n-1] \end{cases}.$$

Since

$$\begin{aligned}
& \left| \{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n - 1 \} \right| \\
&= \left| \{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n - 2 \} \right| = \left| \{ \mathbf{x} \in \mathbb{N}^d \mid \mathbb{1}\mathbf{x} = n - 1 \} \right| \\
&= \alpha_s^d(n - 1) + \alpha_s^{d-1}(n) = \alpha^d(n - 1) + \alpha^{d-1}(n) \\
&= \alpha^d(n),
\end{aligned}$$

this proves the claim.

Lemma 3.2. *For each n the number $\alpha_s^d(n)$ is the number of points in*

$$S_{\alpha_s^d}(n) = \{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n - 1 \}.$$

We now consider the case of ordinary simplexes. Let α^d be a d -simplex with $\text{vert}(\alpha^d) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$. The sets $\text{vert}(\alpha_s^d)$ and $\text{vert}(\alpha^d)$ are affinely independent, thus there is a bijective affine map f_{α^d} such that

$$f_{\alpha^d}(\mathbf{e}_i) = \mathbf{v}_i$$

when $i \in [d + 1]$. It follows that $f_{\alpha^d}(\alpha_s^d) = \alpha^d$. Since f_{α^d} is a bijection, we can correlate $\alpha^d(n)$ to the points in $f_{\alpha^d}(S_{\alpha_s^d}(n))$.

Let α^d be a d -simplex and F be a facet of a d -simplex $\text{conv}(f_{\alpha^d}(S_{\alpha_s^d}(n)))$. Then the number of points in $F \cap f_{\alpha^d}(S_{\alpha_s^d}(n))$ is

$$\alpha^{d-1}(n) = \binom{n + d - 1 - 1}{d - 1}.$$

Now eliminating all points of $F \cap f_{\alpha^d}(S_{\alpha_s^d}(n))$ from $f_{\alpha^d}(S_{\alpha_s^d}(n))$ changes $f_{\alpha^d}(S_{\alpha_s^d}(n))$ into one of

$$S_{\alpha^d}^i(n) = f_{\alpha^d}(S_{\alpha_s^d}^i(n))$$

for $i \in [d + 1]$ where

$$\begin{cases} S_{\alpha_s^d}^i(n) = \{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n - 1, \mathbf{f}_i\mathbf{x} \geq 1 \} \\ \left| S_{\alpha_s^d}^i(n) \right| = \binom{n-1+d-1}{d} \end{cases}.$$

We call this process the *facet-cut* and we represent it by

$$\alpha^d(n) - \alpha^{d-1}(n) = \alpha^d(n - 1).$$

In general, successive k facet-cuts on $\alpha^d(n)$ yield $\alpha^d(n - k)$.

3.4. The geometric description of polytope numbers. We have geometrically described simplex numbers by arranging points in a simplex. By using this description, we consider the geometric description of polytope numbers for a polytope by arranging points in a polytope.

Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P . Assuming that $P(n)$ are the $V(P)$ -polytope numbers, we define

$$\begin{cases} S_P(n) = \bigcup_{\alpha^d \in \mathcal{F}_d(\mathcal{C}_P)} S_{\alpha^d}(n) \\ S_P(n)^\# = S_P(n) - \bigcup_{\substack{\alpha^{d-1} \in \mathcal{F}_{d-1}(\mathcal{C}_P) \\ \alpha^{d-1} \subseteq \partial(P)}} S_{\alpha^{d-1}}(n) \end{cases}.$$

By using double induction on d and n , we show that

$$\begin{cases} P(n) = |S_P(n)| \\ P(n)^\# = |S_P(n)^\#| \end{cases}.$$

Suppose that $d = 0$. By definition,

$$\begin{cases} |S_P(0)| = 0, |S_P(n)| = 1 \text{ for } n \geq 1 \\ |S_P(0)^\#| = 0, |S_P(n)^\#| = 1 \text{ for } n \geq 1 \end{cases}.$$

Assume that $d \geq 1$ and $n \in [1]_0$. Similarly,

$$\begin{cases} |S_P(0)| = 1, |S_P(1)| = 1 \\ |S_P(0)^\#| = 0, |S_P(1)^\#| = 0 \end{cases}.$$

From now on, we assume that $d \geq 2$ and $n \geq 2$.

For a point p and a set S we define $p + S = \{p + s \mid s \in S\}$. By the definition of $S_P(n)$,

$$\begin{aligned} S_P(n) &= (\mathbf{v}_P + S_P(n-1)) \uplus \left\{ S_P(n) \setminus (\mathbf{v}_P + S_P(n-1)) \right\} \\ &= (\mathbf{v}_P + S_P(n-1)) \uplus \left(\bigcup_{\substack{\mathbf{v}_P \notin \alpha^{d-1} \\ \alpha^{d-1} \in \mathcal{F}_{d-1}(\mathcal{C}_P)}} S_{\alpha^{d-1}}(n) \right). \end{aligned}$$

Since \mathcal{C}_P is a pointed triangulation, by induction on d

$$\bigcup_{\substack{\mathbf{v}_P \notin \alpha^{d-1} \\ \alpha^{d-1} \in \mathcal{F}_{d-1}(\mathcal{C}_P)}} S_{\alpha^{d-1}}(n) = \bigcup_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} S_F(n) = \uplus_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} S_F(n)^\#,$$

which yields

$$S_P(n) = (\mathbf{v}_P + S_P(n-1)) \uplus \left(\biguplus_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} S_F(n)^\# \right).$$

Therefore, by double induction on d and n ,

$$\begin{aligned} |S_P(n)| &= \left| (\mathbf{v}_P + S_P(n-1)) \uplus \left(\biguplus_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} S_F(n)^\# \right) \right| \\ &= |S_P(n-1)| + \sum_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} |S_F(n)^\#| = P(n-1) + \sum_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} F(n)^\# \\ &= P(n). \end{aligned}$$

Similarly,

$$\begin{aligned} |S_P(n)^\#| &= \left| S_P(n) \setminus \bigcup_{\substack{\alpha^{d-1} \in \mathcal{F}_{d-1}(C_P) \\ \alpha^{d-1} \subseteq \partial(P)}} S_{\alpha^{d-1}}(n) \right| \\ &= \left| S_P(n) \setminus \bigcup_{F \in \mathcal{F}(P) \setminus \{P\}} S_F(n) \right| = \left| S_P(n) \setminus \biguplus_{F \in \mathcal{F}(P) \setminus \{P\}} S_F(n)^\# \right| \\ &= |S_P(n)| - \sum_{F \in \mathcal{F}(P) \setminus \{P\}} |S_F(n)^\#| = P(n) - \sum_{F \in \mathcal{F}(P) \setminus \{P\}} F(n)^\# \\ &= P(n)^\#. \end{aligned}$$

Theorem 3.3. *Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P . Suppose that $P(n)$ are the $V(P)$ -polytope numbers. Then*

$$\begin{aligned} P(n) &= \left| \bigcup_{\alpha^d \in \mathcal{F}_d(C_P)} S_{\alpha^d}(n) \right|, \\ P(n)^\# &= \left| \left(\bigcup_{\alpha^d \in \mathcal{F}_d(C_P)} S_{\alpha^d}(n) \right) \setminus \left(\bigcup_{\substack{\alpha^{d-1} \in \mathcal{F}_{d-1}(C_P) \\ \alpha^{d-1} \subseteq \partial(P)}} S_{\alpha^{d-1}}(n) \right) \right|. \end{aligned}$$

3.5. The vertex description of polytope numbers. The $V(P)$ -polytope numbers are determined by both the set $V(P)$ and the facial information of P . For example, for each n the number $\alpha_s^d(n)$ is, by the geometric description of polytope numbers, the number of points in the set

$$S_{\alpha_0^d}(n) = \{ \mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n-1 \}.$$

By the way,

$$\begin{aligned} \{\mathbf{x} \in \mathbb{N}^{d+1} \mid \mathbb{1}\mathbf{x} = n-1\} &= \left\{ \sum_{j \in [n-1]} \mathbf{e}_{i_j} \mid 1 \leq i_1 \leq i_2 \leq \dots \leq i_{n-1} \leq d+1 \right\} \\ &= \left\{ \sum_{i \in [n-1]} \mathbf{v}_{F_i} \mid F_i \in \mathcal{F}(\alpha_s^d), F_i \supseteq F_{i+1} \right\}, \end{aligned}$$

hence for each n the number $\alpha_s^d(n)$ equals the number of points in the set

$$\left\{ \sum_{i \in [n-1]} \mathbf{v}_{F_i} \mid F_i \in \mathcal{F}(\alpha_s^d), F_i \supseteq F_{i+1} \right\}.$$

In general, if P is a d -polytope, then for each n the number $P(n)$ is the number of points in the set

$$\bigcup_{\alpha^d \in \mathcal{F}(\mathcal{C}_P)} f_{\alpha^d}(\alpha_s^d),$$

thus it may be possible to describe polytope numbers by vertex sets.

Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P . For $n \geq 0$ we define two sequences of sets $T_P(n)$ and $T_P(n)^\sharp$ by double induction on the numbers d and n .

Let $d = 0$. We define

$$\begin{cases} T_P(0) = T_P(0)^\sharp = \emptyset \\ T_P(1) = T_P(1)^\sharp = \{\mathbf{0}\} \end{cases},$$

and for $n \geq 1$

$$T_P(n) = T_P(n)^\sharp = \{\mathbf{v}_P\}.$$

Let $d \geq 1$. We define

$$\begin{cases} T_P(0) = T_P(0)^\sharp = \emptyset \\ T_P(1) = T_P(1)^\sharp = \{\mathbf{0}\} \end{cases}$$

and

$$\begin{cases} T_P(n) &= \bigcup_{F \in \mathcal{F}(P)} (\mathbf{v}_F + T_F(n-1)) = \left\{ \sum_{i \in [n-1]} \mathbf{v}_{F_i} \mid F_i \supseteq F_{i+1} \right\}, \\ T_P(n)^\sharp &= T_P(n) \setminus \left(\bigcup_{F \in \mathcal{F}(P) \setminus \{P\}} T_F(n) \right) \\ &= \left\{ \sum_{i \in [n-1]} \mathbf{v}_{F_i} \mid F_i \supseteq F_{i+1}, \text{conv}(\{\mathbf{v}_{F_1}, \mathbf{v}_{F_2}, \dots, \mathbf{v}_{F_{n-1}}\}) \subseteq \partial(P) \right\}. \end{cases}$$

for $n \geq 2$.

We claim that

$$(3.4) \quad \begin{cases} |T_P(n)| = P(n) \\ |T_P(n)^\#| = P(n)^\# \end{cases}.$$

If either $d = 0$ or $n = 0$, then the identities in (3.4) are apparent. Suppose that $d \geq 1$ and $n \geq 1$. Then

$$\begin{aligned} T_P(n) &= \bigcup_{F \in \mathcal{F}(P)} (\mathbf{v}_F + T_F(n-1)) \\ &= (\mathbf{v}_P + T_P(n-1)) \uplus \left\{ \bigcup_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} (\mathbf{v}_F + T_F(n-1)) \right\} \\ &= (\mathbf{v}_P + T_P(n-1)) \uplus \left\langle \bigcup_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} \left[T_F(n)^\# \cup \left\{ \bigcup_{\substack{\mathbf{v}_F \in G \\ G \in \mathcal{F}(F) \setminus \{F\}}} (\mathbf{v}_G + T_G(n-1)) \right\} \right] \right\rangle. \end{aligned}$$

Since

$$\begin{aligned} &\bigcup_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} \left[T_F(n)^\# \cup \left\{ \bigcup_{\substack{\mathbf{v}_F \in G \\ G \in \mathcal{F}(F) \setminus \{F\}}} (\mathbf{v}_G + T_G(n-1)) \right\} \right] \\ &= \bigcup_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} T_F(n)^\# = \uplus_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} T_F(n)^\#, \end{aligned}$$

we obtain

$$T_P(n) = (\mathbf{v}_P + T_P(n-1)) \uplus \left(\uplus_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} T_F(n)^\# \right).$$

Therefore, by double induction on d and n ,

$$\begin{aligned} |T_P(n)| &= \left| (\mathbf{v}_P + T_P(n-1)) \uplus \left(\uplus_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} T_F(n)^\# \right) \right| \\ &= |\mathbf{v}_P + T_P(n-1)| + \sum_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} |T_F(n)^\#| \\ &= P(n-1) + \sum_{\substack{\mathbf{v}_P \notin F \\ F \in \mathcal{F}(P)}} F(n)^\# \\ &= P(n). \end{aligned}$$

By the definition of $T_P(n)^\sharp$

$$\begin{aligned} T_P(n)^\sharp &= T_P(n) \setminus \left(\bigcup_{F \in \mathcal{F}(P) \setminus \{P\}} T_F(n) \right) \\ &= T_P(n) \setminus \left(\biguplus_{F \in \mathcal{F}(P) \setminus \{P\}} T_F(n)^\sharp \right), \end{aligned}$$

therefore

$$\begin{aligned} |T_P(n)^\sharp| &= \left| T_P(n) \setminus \left(\biguplus_{F \in \mathcal{F}(P) \setminus \{P\}} T_F(n)^\sharp \right) \right| \\ &= |T_P(n)| - \sum_{F \in \mathcal{F}(P) \setminus \{P\}} |T_F(n)^\sharp| \\ &= P(n) - \sum_{F \in \mathcal{F}(P) \setminus \{P\}} F(n)^\sharp \\ &= P(n)^\sharp. \end{aligned}$$

Theorem 3.4. *For a polytope P if $P(n)$ are the $V(P)$ -polytope numbers, then*

$$\begin{cases} P(0) = P(0)^\sharp = |\emptyset| \\ P(1) = P(1)^\sharp = |\{\mathbf{0}\}| \end{cases}$$

and

$$\begin{cases} P(n) = \left| \left\{ \sum_{i \in [n-1]} \mathbf{v}_{F_i} \mid F_i \supseteq F_{i+1} \right\} \right|, \\ P(n)^\sharp = \left| \left\{ \sum_{i \in [n-1]} \mathbf{v}_{F_i} \mid F_i \supseteq F_{i+1}, \text{conv}(\{\mathbf{v}_{F_1}, \mathbf{v}_{F_2}, \dots, \mathbf{v}_{F_{n-1}}\}) \subseteq \partial(P) \right\} \right|. \end{cases}$$

when $n \geq 2$.

3.5.1. Computations of polytope numbers by the vertex description of polytope numbers. Kim computed polytope numbers for regular polytopes by the definition of polytope numbers [3]. We compute polytope numbers for several polytopes by the vertex description of polytope numbers and show that our computation coincides Kim's computation.

For the standard d -simplex α_s^d we suppose that $V(\alpha_s^d)$ is formed by the linear function

$$L_{\alpha^d}(x_1, x_2, \dots, x_{d+1}) = \sum_{i \in [d+1]} ix_i.$$

If F is a face of α_s^d , then

$$\begin{cases} F = \text{conv}\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_k}\} \\ \mathbf{v}_F = \mathbf{e}_{i_1} \end{cases}$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq d+1$. Thus

$$T_{\alpha_s^d}(n) = \left\{ (x_1, x_2, \dots, x_{d+1}) \in \mathbb{N}^{d+1} \mid \sum_{i \in [d+1]} x_i = n-1 \right\},$$

which yields

$$\alpha^d(n) = \alpha_s^d(n) = |T_{\alpha_s^d}(n)| = \binom{n-1+d}{d}.$$

For the standard d -cross polytope β_s^d let

$$L_{\beta_s^d}(x_1, x_2, \dots, x_d) = \sum_{i \in [d]} ix_i$$

be a linear function that forms $V(\beta_s^d)$. If F is a face of β_s^d , then

$$F = \text{conv}\{a_{i_1}\mathbf{e}_{i_1}, \dots, a_{i_k}\mathbf{e}_{i_k}\}$$

where

$$\begin{cases} \{i_1, i_2, \dots, i_k\} \subseteq [d] \\ \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \subseteq \{1, -1\} \end{cases}.$$

Let I_F^+ and I_F^- be subsets of $\{i_1, i_2, \dots, i_k\}$ defined by

$$\begin{cases} a_i = 1 & \text{for } i \in I_F^+ \\ a_i = -1 & \text{for } i \in I_F^- \end{cases}.$$

For each face F of β_s^d , the apex \mathbf{v}_F is

$$\mathbf{v}_F = \begin{cases} \mathbf{e}_1 & \text{for } F = P \\ \mathbf{e}_{\min(I_F^+)} & \text{for } I_F^+ \neq \emptyset \\ -\mathbf{e}_{\max(I_F^-)} & \text{for } I_F^+ = \emptyset \end{cases}.$$

We define a partial order \prec on $V(\beta_s^d)$ by

$$\mathbf{v}_{F_1} \prec \mathbf{v}_{F_2} \text{ for } F_1 \subseteq F_2.$$

Then

$$\begin{cases} \mathbf{e}_i \prec \mathbf{e}_j, -\mathbf{e}_i \prec -\mathbf{e}_j & \text{if } i \leq j \\ \mathbf{e}_i \succ -\mathbf{e}_j & \text{if } i \neq j \end{cases}.$$

Therefore $\beta^d(n)$ is the number of lattice points $(x_1^+ - x_1^-, x_2, x_3, \dots, x_d)$ satisfying

$$\begin{cases} x_1^+, x_1^- \geq 0 \\ x_1^+ + x_1^- + \sum_{i \in [d] \setminus \{1\}} |x_i| = n - 1 \end{cases}.$$

Recall that a lattice point is a point each of whose coordinates is an integer. We can easily show that the number of such lattice points equals the number of lattice points (x_1, x_2, \dots, x_d) obeying

$$\begin{cases} x_1 \geq 0 \\ x_1 + \sum_{i \in [d] \setminus \{1\}} |x_i| \leq n - 1 \end{cases}.$$

For the standard d -measure polytope γ_s^d let

$$L_{\gamma_s^d}(x_1, x_2, \dots, x_d) = - \left(\sum_{i \in [d]} (d + 1 - i) x_i \right)$$

be a linear function in general position for γ_s^d that forms $V(\gamma_s^d)$. A face F of γ^d is determined by the intersection of the following hyperplanes that defines facets of γ_s^d ;

$$x_{i_1} = a_{i_1}, x_{i_2} = a_{i_2}, \dots, x_{i_k} = a_{i_k}$$

where

$$\begin{cases} \{i_1, i_2, \dots, i_k\} \subseteq [d] \\ \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \subseteq [1]_0 \end{cases},$$

which gives $\mathbf{v}_F = \sum_{j \in [k]} a_{i_j} \mathbf{e}_{i_j}$. Thus

$$\gamma^d(n) = \gamma_s^d(n) = \left| \{(x_1, x_2, \dots, x_d) \mid x_i \in [n - 1]_0\} \right|.$$

Let

$$\alpha_k^{d-1} = \left\{ (x_1, x_2, \dots, x_d) \mid \sum_{i \in [d]} x_i = k, x_i \geq 0 \right\}$$

be a hypersimplex and

$$L_{\alpha_k^{d-1}}(x_1, x_2, \dots, x_d) = \sum_{i \in [d]} (d+1-i)x_i$$

be a linear function in general position for α_k^{d-1} that forms $V(\alpha_k^{d-1})$. Since the hyperplanes $x_i = 0$ and $x_i = 1$ for $i \in [d]$ determine the facets of α_k^{d-1} , the hyperplanes

$$x_{i_1} = a_{i_1}, x_{i_2} = a_{i_2}, \dots, x_{i_k} = a_{i_k}$$

satisfying

$$\begin{cases} \{i_1, i_2, \dots, i_k\} \subseteq [d] \\ \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \subseteq [1]_0 \end{cases},$$

determine a face F of α_k^{d-1} . Let

$$\begin{cases} I_F^0 = \{i \in \{i_1, \dots, i_k\} \mid a_i = 0\} \\ I_F^1 = \{i \in \{i_1, i_2, \dots, i_k\} \mid a_i = 1\} \end{cases}$$

and I_F be the set of the first $k - |I_F^1|$ minimal indexes in

$$[d] \setminus (I_F^0 \cup I_F^1).$$

Note that $|I_F^0| \leq d - k$ and $|I_F^1| \leq k$. Then

$$\mathbf{v}_F = \sum_{i \in I_F^1} \mathbf{e}_i + \sum_{i \in I_F} \mathbf{e}_i.$$

We claim that

$$\alpha_k^{d-1}(n) = \left| \left\{ (x_1, x_2, \dots, x_d) \mid \sum_{i \in [d]} x_i = k(n-1), x_i \in [n-1]_0 \right\} \right|.$$

To establish this claim, we need only show that for each (x_1, x_2, \dots, x_n) with

$$\begin{cases} \sum_{i \in [d]} x_i = k(n-1) \\ x_i \in [n-1]_0 \end{cases}$$

there is a set $\{\mathbf{v}_{F_1}, \mathbf{v}_{F_2}, \dots, \mathbf{v}_{F_{n-1}}\}$ in $V(P)$ such that

$$\begin{cases} F_i \supseteq F_{i+1} \\ \sum_{i \in [n-1]} \mathbf{v}_{F_i} = (x_1, x_2, \dots, x_d) \end{cases}.$$

Let $\mathbf{x}_1 = (x_{1,1}, x_{1,2}, \dots, x_{1,d})$ with

$$\begin{cases} \sum_{i \in [d]} x_{1,i} = k(n-1) \\ x_{1,i} \in [n-1]_0 \end{cases}.$$

We define I_1 to be the set of indexes i such that $x_{1,i} = n-1$, J_1 to be the set of $k - |I_1|$ smallest numbers from $[d] - I_1$, and

$$\begin{cases} \mathbf{y}_1 = \sum_{i \in I_1 \cup J_1} \mathbf{e}_i \\ \mathbf{x}_2 = \mathbf{x}_1 - \mathbf{y}_1 \end{cases}.$$

Inductively, we define I_l to be the set of indexes i such that $x_{l,i} = n-l$, J_l to be the set of $k - |I_l|$ smallest numbers from $[d] - I_l$, and

$$\begin{cases} \mathbf{y}_l = \sum_{i \in I_l \cup J_l} \mathbf{e}_i \\ \mathbf{x}_{l+1} = \mathbf{x}_l - \mathbf{y}_l \end{cases}.$$

We can easily show that $\mathbf{x}_1 = \sum_{i \in [n-1]} \mathbf{y}_i$ and there are faces F_1, F_2, \dots, F_{n-1} of α_k^{d-1} such that

$$\begin{cases} F_i \supseteq F_{i+1} \\ \mathbf{v}_{F_i} = \mathbf{x}_i \end{cases}.$$

4. DECOMPOSITION THEOREMS FOR POLYTOPE NUMBERS

We have defined pointed triangulation in Section 2 and the geometric description of polytope numbers in Section 3. By combining these concepts, we develop several methods to represent polytope numbers by sums of simplex numbers, called decomposition theorems. We also investigate relations between these decomposition theorems.

4.1. Decomposition theorem 1. Decomposition theorem 1 shows that every d -polytope numbers can be decomposed into d -simplex numbers.

Theorem 4.1 (Decomposition theorem 1). *Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P . Suppose that $P(n)$ are the $V(P)$ -polytope*

numbers. Then there are nonnegative integers a_1, a_2, \dots, a_{d-1} such that

$$P(n) = \alpha^d(n) + \sum_{i \in [d-1]} a_i \alpha^d(n-i).$$

Proof. Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P . Then the geometric description of polytope numbers furnishes $P(n) = |S_P(n)|$ where

$$S_P(n) = \bigcup_{\alpha^d \in \mathcal{F}_d(\mathcal{C}_P)} S_{\alpha^d}(n).$$

Let $\mathcal{F}_d(\mathcal{C}_P) = \{\alpha_1^d, \alpha_2^d, \dots, \alpha_s^d\}$. Using Lemma 2.2, we can assign a shelling to $\mathcal{F}_d(\mathcal{C}_P)$ that satisfies the conditions of Lemma 2.2. We assume that the indexes of elements in $\mathcal{F}_d(\mathcal{C}_P)$ equal those in the proof of Lemma 2.2.

Let

$$\begin{cases} \mathcal{P}_1 = \{S_{\alpha_1^d}(n)\} \\ \mathcal{Q}_1 = S_{\alpha_1^d}(n) \end{cases}.$$

Then

$$|\mathcal{Q}_1| = \alpha^d(n).$$

For $k \geq 2$ defining

$$\begin{cases} \mathcal{P}_k = \mathcal{P}_{k-1} \cup \{S_{\alpha_k^d}(n)\} \\ \mathcal{Q}_k = \mathcal{Q}_{k-1} \cup S_{\alpha_k^d}(n) \end{cases},$$

we inductively suppose that the number of facets of $\text{conv}(S_{\alpha_k^d}(n))$ in

$$\bigcup_{\alpha^d \in \mathcal{P}_{k-1}} \text{conv}(S_{\alpha^d}(n)),$$

denoted by l_k , satisfies $l_k \in [d-1]$. Thus we need to eliminate every point on those l_k facets from $\text{conv}(S_{\alpha_k^d}(n))$ to compute $|\mathcal{Q}_k|$. For this elimination, we apply successive l_k facet-cuts to $S_{\alpha_k^d}(n)$. Then

$$|\mathcal{Q}_k| = \alpha^d(n) + \sum_{i \in [k-1]} \alpha^d(n - l_{i+1}).$$

Continuing this operation until $k = m$ yields

$$|\mathcal{Q}_m| = \alpha^d(n) + \sum_{i \in [s-1]} \alpha^d(n - l_{i+1}).$$

Since $l_k \in [d-1]$ when $k \in [s] \setminus \{1\}$, there are nonnegative integers a_1, a_2, \dots, a_{d-1} such that

$$\alpha^d(n) + \sum_{i \in [s-1]} \alpha^d(n - l_{i+1}) = \alpha^d(n) + \sum_{i \in [d-1]} a_i \alpha^d(n - i).$$

By $\mathcal{Q}_m = S_P(n)$,

$$P(n) = \alpha^d(n) + \sum_{i \in [d-1]} a_i \alpha^d(n - i).$$

□

Remark 4.1. For a d -polytope P with the $V(P)$ -pointed triangulation \mathcal{C}_P , the number $P(n)$ is a polynomial in n and it can be determined by d different values of n . Thus, whichever shelling of \mathcal{C}_P we may choose, the pointed triangulation \mathcal{C}_P uniquely determines the coefficients a_1, a_2, \dots, a_{d-1} . This means that a polynomial identity of polytope numbers explains a geometric property of polytopes.

4.2. Decomposition theorem 2. Decomposition theorem 2 shows that we can describe polytope numbers for a polytope by the facial information of a pointed triangulation.

Theorem 4.2 (Decomposition theorem 2). *Let \mathcal{C}_P be the $V(P)$ -pointed triangulation. If $P(n)$ are the $V(P)$ -polytope numbers, then*

$$P(n) = \sum_{i \in [d-1]} (-1)^{d-i} b_i \alpha^i(n)$$

where b_i is the number of i -simplexes α^i in \mathcal{C}_P such that

$$\begin{cases} \mathbf{v}_P \in \alpha^i \\ \alpha^i \cap \text{int}(P) \neq \emptyset \end{cases}.$$

Proof. Let \mathcal{C}_P be the $V(P)$ -pointed triangulation. We assume that the order on d -simplexes in \mathcal{C}_P is the same as that in Decomposition theorem 1. We can naturally endow the $V(P_k)$ -pointed triangulation of

$$P_k = \bigcup_{i \in [k]} \text{conv}(S_{\alpha_i^d})$$

from \mathcal{C}_P for $k \in [s]$. Even when P_k is not convex, this triangulation is possible by the definition of pointed triangulation. We denote by \mathcal{C}_{P_k} the endowed $V(P_k)$ -pointed triangulation of P_k .

Let $S_{P_k}(n) = \bigcup_{i \in [k]} S_{\alpha_i^d}(n)$. If $k = 1$ then

$$|S_{P_1}(n)| = \alpha^d(n).$$

Inductively, we assume that

$$|S_{P_k}(n)| = \sum_{i \in [d]} (-1)^{d-i} b_{k,i} \alpha_i^d(n),$$

where $b_{k,i}$ is the number of $\alpha^i \in \mathcal{C}_{P_k}$ such that

$$\begin{cases} \mathbf{v}_P \in \alpha^i \\ \alpha^i \cap \text{int}\left(\bigcup_{j \in [k]} \alpha_j^d\right) \neq \emptyset \end{cases}.$$

We claim that

$$(4.1) \quad |S_{P_{k+1}}(n)| = \sum_{i \in [d]} (-1)^{d-i} b_{k+1,i} \alpha_i^d(n).$$

If either $d = 0$ or $d = 1$, then the identity (4.1) is obvious. Suppose that $d \geq 2$. For

$$F_{k+1}^{d-1} = \alpha_{k+1}^d \cap |\mathcal{C}_{P_k}|$$

let $\mathcal{C}_{F_{k+1}^{d-1}}$ be the endowed $V(F_{k+1}^{d-1})$ -pointed triangulation of F_{k+1}^{d-1} from \mathcal{C}_{P_k} .

By the definition of pointed triangulation, there are $(d-2)$ -simplexes

$$\alpha_{k+1,1}^{d-2}, \alpha_{k+1,2}^{d-2}, \dots, \alpha_{k+1,l_{k+1}}^{d-2}$$

such that

$$F_{k+1}^{d-1} = \bigcup_{i \in [l_{k+1}]} \alpha_{k+1,i}^{d-2}.$$

Thus $\mathcal{C}_{F_{k+1}^{d-1}}$ is the $V(F_{k+1}^{d-1})$ -pointed triangulation of F_{k+1}^{d-1} . By induction on d

$$\left| \bigcup_{i \in [l_{k+1}]} S_{\alpha_{k+1,i}^{d-1}}(n) \right| = \sum_{i \in [d-1]} (-1)^{d-i} c_{k+1,i} \alpha^i(n),$$

where $c_{k+1,i}$ is the number of i -simplexes α^i in $\mathcal{C}_{F_{k+1}^{d-1}}$ such that

$$\begin{cases} \mathbf{v}_P \in \alpha^i \\ \alpha^i \cap \text{int}(F_{k+1}^{d-1}) \neq \emptyset \end{cases}.$$

Hence

$$\begin{aligned} |S_{P_{k+1}}(n)| &= |S_{P_k}(n)| + |\alpha_{k+1}^d(n)| - \left| \bigcup_{i \in [k+1]} \alpha_{k+1,i}^{d-1}(n) \right| \\ &= \sum_{i \in [d]} (-1)^{d-i} b_{k,i} \alpha^i(n) + \alpha^d(n) - \sum_{i \in [d-1]} (-1)^{d-1-i} c_{k+1,i} \alpha^i(n) \\ &= (b_{k,d} + 1) \alpha^d(n) + \sum_{i \in [d-1]} (-1)^{d-i} (b_{k,i} + c_{k+1,i}) \alpha^i(n). \end{aligned}$$

Since $b_{k,d} + 1$ is the number of d -simplexes α^d in $\mathcal{C}_{P_{k+1}}$ such that

$$\begin{cases} \mathbf{v}_P \in \alpha^d \\ \alpha^d \cap \text{int}\left(\bigcup_{j \in [k+1]} \alpha_{k+1,j}^d\right) \neq \emptyset \end{cases}$$

and for $i \in [d-1]$ the number $b_{k,i} + c_{k+1,i}$ is the number of i -simplexes α^i in \mathcal{C}_{k+1} such that

$$\begin{cases} \mathbf{v}_P \in \alpha^i \\ \alpha^i \cap \text{int}\left(\bigcup_{i \in [k+1]} \alpha_{k+1,i}^d\right) \neq \emptyset \end{cases},$$

we prove our assertion.

If we continue this process until $k = s$, then

$$P(n) = \sum_{i \in [d]} (-1)^{d-i} b_i \alpha^i(n)$$

where b_i is the number of α^i in P such that

$$\begin{cases} \mathbf{v}_P \in \alpha^i \\ \alpha^i \cap \text{int}(P) \neq \emptyset \end{cases}.$$

□

4.3. Decomposition theorem 3. The main idea of Decomposition theorem 3 is to represent a pointed triangulation by a disjoint union and then to apply this representation to polytope numbers.

Theorem 4.3 (Decomposition theorem 3-1). *Let \mathcal{C}_P be the $V(P)$ -pointed triangulation and*

$$\mathcal{F}_k(\mathcal{C}_P, \mathbf{v}_P) = \{\alpha^k \mid \mathbf{v}_P \in \alpha^k, \alpha^k \in \mathcal{F}_k(\mathcal{C}_P)\}.$$

Then the $V(P)$ -polytope numbers are represented by

$$P(n) = \sum_{k \in [d]} c_k \alpha^k (n - k)$$

where $c_k = |\mathcal{F}_k(\mathcal{C}_P, \mathbf{v}_P)|$.

Proof. For each $\alpha^k \in \mathcal{F}_k(\mathcal{C}_P, \mathbf{v}_P)$ there are exactly k facets α^{k-1} of α^k that are in $\mathcal{F}_{k-1}(\mathcal{C}_P, \mathbf{v}_P)$ such that $\mathbf{v}_P \in \alpha^{k-1}$. Correlating $\alpha^k(n)$ to $S_{\alpha^k}(n)$, we apply successive k facet-cuts to $S_{\alpha^k}(n)$. Then these successive facet-cuts change $S_{\alpha^k}(n)$ into $S_{\alpha^k}^k(n)$ where

$$|S_{\alpha^k}^k(n)| = \alpha^k(n - k).$$

Each pair of distinct faces α^i and α^j of \mathcal{C}_P contained in $\mathcal{F}_i(\mathcal{C}_P, \mathbf{v}_P)$ satisfies

$$S_{\alpha^i}^i(n) \cap S_{\alpha^j}^j(n) = \emptyset,$$

thus

$$\begin{aligned} P(n) &= \sum_{k \in [d]} \sum_{\alpha^k \in \mathcal{F}_k(\mathcal{C}_P, \mathbf{v}_P)} S_{\alpha^k}^k(n) \\ &= \sum_{k \in [d]} c_k \alpha^k (n - k). \end{aligned}$$

□

Instead of considering all of the faces in \mathcal{C}_P , if we consider only those faces of \mathcal{C}_P that have nonempty intersection with $\text{int}(P)$, then we obtain the following corollary.

Corollary 4.4 (Decomposition theorem 3-2). *Let*

$$\begin{cases} \mathcal{F}_{-1}^1(\mathcal{C}_F, \mathbf{v}_P) = \emptyset \\ \mathcal{F}_0^1(\mathcal{C}_F, \mathbf{v}_P) = \{\mathbf{v}_P\} \end{cases},$$

and for $k \in [d]$ let

$$\mathcal{F}_k^1(\mathcal{C}_F, \mathbf{v}_P) = \left\{ \alpha^k \mid \alpha^k \in \mathcal{F}_k(\mathcal{C}_P, \mathbf{v}_P), \alpha^k \cap \text{int}(P) \neq \emptyset \right\}.$$

For each $\alpha^k \in \mathcal{F}_k^1(\mathcal{C}_F, \mathbf{v}_P)$ with $k \in [d]_0$ let $f(\alpha^k)$ be the number of facets of α^k that are in $\mathcal{F}_{k-1}^1(\mathcal{C}_F, \mathbf{v}_P)$. Then the $V(P)$ -polytope numbers are represented by

$$P(n) = \sum_{k \in [d]_0} \sum_{\alpha^k \in \mathcal{F}_k^1(\mathcal{C}_F, \mathbf{v}_P)} \alpha^k (n - f(\alpha^k)).$$

4.4. Decomposition theorem 4. Let \mathcal{C}_P be a pointed triangulation of P . Then

$$P = \bigsqcup_{k \in [d]_0} \bigsqcup_{\alpha^k \in \mathcal{F}_k(\mathcal{C}_P)} \text{relint}(\alpha^k).$$

Using this decomposition, we consider Decomposition theorem 4.

Theorem 4.5 (Decomposition theorem 4). *Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P and for $k \in [d]_0$ let d_k be the number of k -simplexes in \mathcal{C}_P . Then the $V(P)$ -polytope numbers are represented by*

$$P(n) = \sum_{k \in [d]} d_k \alpha^k (n - (k + 1)).$$

Proof. Each pair of two distinct faces α^{k_1} and α^{k_2} of \mathcal{C}_P satisfies

$$\text{relint}(\alpha^{k_1}) \cap \text{relint}(\alpha^{k_2}) = \emptyset,$$

thus

$$(4.2) \quad P = \bigsqcup_{k \in [d]_0} \bigsqcup_{\alpha^k \in \mathcal{C}_P} \text{relint}(\alpha^k).$$

The identity (4.2) yields

$$S_P(n) = \bigsqcup_{k \in [d]_0} \bigsqcup_{\alpha^k \in \mathcal{F}(\mathcal{C}_P)} S_{\alpha^k}(n)^\sharp.$$

Since the interior k -simplex numbers $\alpha^k(n)^\sharp$ satisfy

$$\alpha^k(n)^\sharp = \alpha^k(n - (k + 1)),$$

if we denote $d_k = |\mathcal{F}_k(\mathcal{C}_P)|$ then

$$\begin{aligned}
 P(n) &= |S_P(n)| = \left| \biguplus_{k \in [d]_0} \biguplus_{\alpha^k \in \mathcal{F}(\mathcal{C}_P)} S_{\alpha^k}(n)^\# \right| \\
 &= \sum_{k \in [d]_0} \sum_{\alpha^k \in \mathcal{F}_k(\mathcal{C}_P)} \alpha^k(n - (k + 1)) \\
 &= \sum_{k \in [d]_0} d_k \alpha^k(n - (k + 1)).
 \end{aligned}$$

□

4.5. Relations between decomposition theorems. Decomposition theorems provides several methods to decompose polytope numbers into simplex numbers. Thus relations between decomposition theorems may exist. We consider such relations in this subsection.

4.5.1. Decomposition theorem 1 and other decomposition theorems. Decomposition theorem 1 represents d -polytope numbers by sums of d -simplex numbers, whereas other decomposition theorems represent d -polytope numbers by sums of various dimensional simplex numbers. Thus we use the relation $\alpha^{d-1}(n) = \alpha^d(n) - \alpha^d(n - 1)$ to derive Decomposition theorem 1 from other decomposition theorems.

The equation $\alpha^{d-1}(n) = \alpha^d(n) - \alpha^d(n - 1)$ produces

$$(4.3) \quad \alpha^i(n) = \sum_{j \in [d-i]_0} (-1)^j \binom{d-i}{j} \alpha^d(n-j),$$

$$(4.4) \quad \alpha^k(n-k) = \sum_{j \in [d-k]_0} (-1)^j \binom{d-k}{j} \alpha^d(n-k-j),$$

$$(4.5) \quad \alpha^k(n - (k + 1)) = \sum_{j \in [d-k]_0} (-1)^j \binom{d-k}{j} \alpha^d(n - (k + 1) - j).$$

Therefore for a d -polytope P the identities (4.3)–(4.5) change decomposition forms of $P(n)$ in Decomposition theorems 2, 3-1, and 4 into

$$\begin{aligned}
P(n) &= \sum_{i \in [d]} (-1)^{d-i} b_i a^i(n) \\
&= \sum_{i \in [d]} (-1)^{d-i} b_i \left(\sum_{j \in [d-i]} (-1)^j \binom{d-i}{j} \alpha^d(n-j) \right) \\
&= \sum_{j \in [d-1]_0} \left(\sum_{i \in [d-j]} (-1)^{d-i-j} b_i \binom{d-i}{j} \right) \alpha^d(n-j),
\end{aligned}$$

$$\begin{aligned}
P(n) &= \sum_{k \in [d]_0} c_k \alpha^k(n-k) \\
&= \sum_{k \in [d]_0} c_k \left(\sum_{j \in [d-k]_0} (-1)^j \binom{d-k}{j} \alpha^d(n-k-j) \right) \\
&= \sum_{l \in [d]_0} \left(\sum_{k \in [l]_0} c_k (-1)^{l-k} \binom{d-k}{l-k} \right) \alpha^d(n-l),
\end{aligned}$$

and

$$\begin{aligned}
P(n) &= \sum_{k \in [d]_0} d_k \alpha^k(n-(k+1)) \\
&= \sum_{k \in [d]_0} d_k \left(\sum_{j \in [d-k]_0} (-1)^j \binom{d-k}{j} \alpha^d(n-(k+1)-j) \right) \\
&= \sum_{l \in [d+1]} \left(\sum_{k \in [l-1]_0} d_k (-1)^{l-1-k} \binom{d-k}{l-1-k} \right) \alpha^d(n-l),
\end{aligned}$$

respectively.

4.5.2. Decomposition theorems 2 and 3-2. For a polytopal complex \mathcal{C}_P , Decomposition theorems 2 and 3-2 use the same facial information of \mathcal{C}_P . Hence we can expect a relation between these two decomposition theorems. We verify such a relation here.

For a finite poset \mathcal{P} we define the *zeta function* ζ of \mathcal{P} to be

$$\zeta(x, y) = 1 \text{ for each } \{x, y\} \subseteq \mathcal{P} \text{ with } x \leq y.$$

Let the inverse function of ζ be μ called the *Möbius function* of \mathcal{P} . Then

$$\mu(x, y) = \begin{cases} 1 & \text{when } x = y \\ - \sum_{x \leq z < y} \mu(x, z) & \text{when } x < y \end{cases}.$$

Theorem 4.6 (Möbius inversion formula [4]). *Let \mathcal{P} be a finite poset and $f : \mathcal{P} \rightarrow \mathbb{C}$ and $g : \mathcal{P} \rightarrow \mathbb{C}$. Then*

$$g(x) = \sum_{y \leq x} f(y) \text{ when } x \in \mathcal{P}$$

if and only if

$$f(x) = \sum_{y \leq x} g(y) \mu(y, x) \text{ when } x \in \mathcal{P}.$$

Theorem 4.7 ([4]). *Let \mathcal{C} be a polytopal complex and $\mathcal{P} = \mathcal{P}(\mathcal{C})$ be the poset on $(\mathcal{C} \setminus \{\emptyset\}) \cup \{|\mathcal{C}|\}$, ordered by*

$$F_i \leq F_j \text{ if } F_i \subseteq F_j.$$

Then

$$\mu_{\mathcal{P}}(F_1, F_2) = \begin{cases} 0, & \text{if } F_2 = |\mathcal{C}| \text{ and } F_1 \text{ lies on the} \\ & \text{boundary of } |\mathcal{C}| \\ (-1)^{\text{rank}(F_2) - \text{rank}(F_1)}, & \text{otherwise} \end{cases}.$$

Let P be a d -polytope with the $V(P)$ -pointed triangulation \mathcal{C}_P and $\mathcal{C}_{P^{d-1}}$ be the polytopal complex formed by the faces of \mathcal{C}_P that do not contain \mathbf{v}_P . We define

$$(4.6) \quad \begin{cases} \mathcal{C}_{P^{d-1}}(n)^{\sharp} = 0 \\ \mathcal{C}_{P^{d-1}}(n) = \mathcal{C}_{P^{d-1}}(n)^{\sharp} + \sum_{k \in [d-1]_0} \sum_{\alpha^k \in \mathcal{C}_{P^{d-1}}} \alpha^k(n)^{\sharp} \end{cases}.$$

Note that the identity (4.6) is a restatement of the decomposition form of Decomposition theorem 3-2. If we let $\mathcal{P} = \mathcal{P}(\mathcal{C}_{P^{d-1}})$, then Theorems 4.6 and 4.7 supply

$$\mathcal{C}_{P^{d-1}}(n)^{\sharp} = \mathcal{C}_{P^{d-1}}(n) + \sum_{k \in [d-1]_0} (-1)^{d-k} b_{k+1} \alpha^k(n)$$

where b_{k+1} is the number of $(k+1)$ -simplexes that contain \mathbf{v}_P and are not on the boundary of P . Since $\mathcal{C}_{P^{d-1}}(n)^\sharp = 0$, we obtain

$$\mathcal{C}_{P^{d-1}}(n) = \sum_{k \in [d-1]_0} (-1)^{d-1-k} b_{k+1} \alpha^k(n).$$

Moreover,

$$\begin{cases} P(n) = \sum_{i \in [n]} \mathcal{C}_{P^{d-1}}(i) \\ \alpha^{k+1}(n) = \sum_{i \in [n]} \alpha^k(i) \end{cases}$$

by definition, thus

$$P(n) = \sum_{k \in [d]} (-1)^{d-k} b_k \alpha^k(n),$$

which is the decomposition form of Decomposition theorem 2.

Instead of triangulation, we consider the polytopal complex $\mathcal{C}_{Q^{d-1}}$ formed by the faces of P that do not contain \mathbf{v}_P . We define

$$\begin{cases} \mathcal{C}_{Q^{d-1}}(n)^\sharp = 0 \\ \mathcal{C}_{Q^{d-1}}(n) = \mathcal{C}_{Q^{d-1}}(n)^\sharp + \sum_{k \in [d-1]_0} \sum_{F^k \in \mathcal{C}_{Q^{d-1}}} F^k(n)^\sharp \end{cases}.$$

The method used in the triangulation case yields

$$\mathcal{C}_{Q^{d-1}}(n) = \sum_{k \in [d-1]} (-1)^{d-1-k} \sum_{F^k \in \mathcal{C}_{Q^{d-1}}} F^k(n).$$

Since $P(n) = \sum_{i \in [n]} \mathcal{C}_{Q^{d-1}}(i)$, if we define $F^{k,s}(n) = \sum_{i \in [n]} F^k(i)$ then

$$P(n) = \sum_{k \in [d]} (-1)^{d-k} \sum_{F^k \in \mathcal{C}_{Q^{d-1}}} F^{k,s}(n).$$

4.6. Computations of coefficients in decomposition theorems. Let P be a d -polytope. By the definition of polytope numbers, for each n the number $P(n)$ is a polynomial of n . Therefore a finite number of values of $P(n)$ allow us to compute the coefficients in decomposition forms of $P(n)$ in decomposition theorems. We perform such computations.

By Decomposition theorem 1

$$P(n) = \sum_{i \in [d-1]} a_i \alpha^d(n-i),$$

thus

$$\begin{pmatrix} P(1) \\ P(2) \\ \dots \\ P(d) \end{pmatrix} = \begin{pmatrix} \alpha^d(1) & 0 & \dots & 0 \\ \alpha^d(2) & \alpha^d(1) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha^d(d) & \alpha^d(d-1) & \dots & \alpha^d(1) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{d-1} \end{pmatrix}.$$

If we let

$$A = \begin{pmatrix} \alpha^d(1) & 0 & \dots & 0 \\ \alpha^d(2) & \alpha^d(1) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha^d(d) & \alpha^d(d-1) & \dots & \alpha^d(1) \end{pmatrix},$$

then the Gaussian elimination produces

$$A^{-1} = \begin{pmatrix} (-1)^0 \binom{d+1}{0} & 0 & \dots & 0 \\ (-1)^1 \binom{d+1}{1} & (-1)^0 \binom{d+1}{0} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (-1)^{d-1} \binom{d+1}{d-1} & (-1)^{d-2} \binom{d+1}{d-2} & \dots & (-1)^0 \binom{d+1}{0} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{d-1} \end{pmatrix} = \begin{pmatrix} (-1)^0 \binom{d+1}{0} & 0 & \dots & 0 \\ (-1)^1 \binom{d+1}{1} & (-1)^0 \binom{d+1}{0} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (-1)^{d-1} \binom{d+1}{d-1} & (-1)^{d-2} \binom{d+1}{d-2} & \dots & (-1)^0 \binom{d+1}{0} \end{pmatrix} \begin{pmatrix} P(1) \\ P(2) \\ \dots \\ P(d) \end{pmatrix}.$$

By Decomposition theorem 2

$$P(n) = \sum_{i \in [d]} (-1)^{d-i} b_i \alpha^i(n),$$

which gives

$$\begin{pmatrix} P(1) \\ P(2) \\ \dots \\ P(d) \end{pmatrix} = \begin{pmatrix} \alpha^1(1) & \alpha^2(1) & \dots & \alpha^d(1) \\ \alpha^1(2) & \alpha^2(2) & \dots & \alpha^d(2) \\ \dots & \dots & \dots & \dots \\ \alpha^1(d) & \alpha^2(d) & \dots & \alpha^d(d) \end{pmatrix} \begin{pmatrix} (-1)^{d-1} b_1 \\ (-1)^{d-2} b_2 \\ \dots \\ (-1)^0 b_d \end{pmatrix}.$$

If we denote

$$B = \begin{pmatrix} \alpha^1(1) & \alpha^2(1) & \cdots & \alpha^d(1) \\ \alpha^1(2) & \alpha^2(2) & \cdots & \alpha^d(2) \\ \cdots & \cdots & \cdots & \cdots \\ \alpha^1(d) & \alpha^2(d) & \cdots & \alpha^d(d) \end{pmatrix}$$

then

$$\begin{pmatrix} (-1)^{d-1}b_1 \\ (-1)^{d-2}b_2 \\ \cdots \\ (-1)^1b_{d-1} \\ (-1)^0b_d \end{pmatrix} = B^{-1} \begin{pmatrix} P(1) \\ P(2) \\ \cdots \\ P(d-1) \\ P(d) \end{pmatrix}$$

where the matrix B^{-1} is

$$\begin{pmatrix} (-1)^0\left(\binom{1}{0} + \binom{2}{1}\right) & (-1)^{-1}\binom{2}{0} & 0 & \cdots & 0 \\ (-1)^1\left(\binom{2}{1} + \binom{3}{2}\right) & (-1)^0\left(\binom{2}{0} + \binom{3}{1}\right) & (-1)^{-1}\binom{3}{0} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{d-2}\left(\binom{d-1}{d-2} + \binom{d}{d-1}\right) & \cdots & \cdots & (-1)^0\left(\binom{d-1}{0} + \binom{d}{1}\right) & (-1)^{-1}\binom{d}{0} \\ (-1)^{d-1}\binom{d}{d-1} & \cdots & \cdots & (-1)^1\binom{d}{1} & (-1)^0\binom{d}{0} \end{pmatrix}.$$

Decomposition theorem 3 yields

$$P(n) = \sum_{i \in [d]} c_i \alpha^d(n-i).$$

Similarly, if we write

$$\begin{pmatrix} P(1) \\ P(2) \\ \cdots \\ P(d+1) \end{pmatrix} = \begin{pmatrix} \alpha^0(1) & 0 & \cdots & 0 \\ \alpha^0(2) & \alpha^1(1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha^0(d+1) & \alpha^1(d) & \cdots & \alpha^d(1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \cdots \\ c_d \end{pmatrix}$$

then

$$\begin{pmatrix} c_0 \\ c_1 \\ \cdots \\ c_d \end{pmatrix} = \begin{pmatrix} (-1)^0\binom{0}{0} & 0 & \cdots & 0 \\ (-1)^1\binom{1}{1} & (-1)^0\binom{1}{0} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ (-1)^d\binom{d}{d} & (-1)^{d-1}\binom{d}{d-1} & \cdots & (-1)^0\binom{d}{0} \end{pmatrix} \begin{pmatrix} P(1) \\ P(2) \\ \cdots \\ P(d+1) \end{pmatrix}.$$

5. ILLUSTRATIONS OF DECOMPOSITION THEOREMS

We apply decomposition theorems to both regular polytopes and the product of simplexes. This application gives new interpretations of some known combinatorial identities and derives new combinatorial identities.

5.1. Decomposition theorem 1.

5.1.1. *Cross polytope* β^d . For fixed numbers $a_i \in \{1, -1\}$ let $\alpha^d(a_1, a_2, \dots, a_{d-1})$ be the d -simplex with the vertex set

$$\{\mathbf{e}_d, -\mathbf{e}_d, a_1\mathbf{e}_1, a_2\mathbf{e}_2, \dots, a_{d-1}\mathbf{e}_{d-1}\},$$

and let $\mathcal{S}(\beta_s^d)$ be the set of such d -simplexes. Then, by a simple reasoning,

$$\beta^d = \bigcup_{\alpha^d \in \mathcal{S}(\beta^d)} \alpha^d.$$

Moreover, the d -simplexes in $\mathcal{S}(\beta_s^d)$ form a pointed triangulation $\mathcal{C}_{\beta_s^d}$ such that

$$\begin{cases} |\mathcal{C}_{\beta_s^d}| = \beta^d \\ \mathcal{C}_{\beta_s^d} = \bigoplus_{\alpha^d \in \mathcal{S}(\beta^d)} \alpha^d \end{cases}.$$

We call $\mathcal{C}_{\beta_s^d}$ the *standard pointed triangulation* of β_s^d and assume that $\beta_s^d(n)$ are formed by $\mathcal{C}_{\beta_s^d}$.

For $i \in [d-1]_0$ let $\mathcal{S}_i(\beta_s^d)$ be the set of $\alpha^d(a_1, a_2, \dots, a_{d-1})$ in $\mathcal{S}(\beta_s^d)$ such that the number of a_j satisfying $a_j = -1$ is i . Then for each i the set $\mathcal{S}_i(\beta_s^d)$ satisfies

$$\begin{cases} |\mathcal{S}_i(\beta_s^d)| = \binom{d-1}{i} \\ \mathcal{S}(\beta_s^d) = \bigsqcup_{i \in [d-1]} \mathcal{S}_i(\beta_s^d) \end{cases}.$$

We say that a d -simplex $\alpha^d(a_1^1, a_2^1, \dots, a_{d-1}^1)$ in $\mathcal{S}(\beta_s^d)$ is *adjacent* to another d -simplex $\alpha^d(a_1^2, a_2^2, \dots, a_{d-1}^2)$ in $\mathcal{S}(\beta_s^d)$ if

$$(a_1^2, a_2^2, \dots, a_{d-1}^2) - (a_1^1, a_2^1, \dots, a_{d-1}^1) = 2\mathbf{e}_j$$

for some $j \in [d-1]$. By the definition of adjacency

$$\dim(\alpha^d(a_1^1, a_2^1, \dots, a_{d-1}^1) \cap \alpha^d(a_1^2, a_2^2, \dots, a_{d-1}^2)) = d-1,$$

and if $\alpha^d(a_1^1, a_2^1, \dots, a_{d-1}^1) \in \mathcal{S}_i(\beta_s^d)$ then $\alpha^d(a_1^2, a_2^2, \dots, a_{d-1}^2) \in \mathcal{S}_{i+1}(\beta_s^d)$. In addition, if $\alpha^d(a_1, a_2, \dots, a_{d-1}) \in \mathcal{S}_i(\beta_s^d)$ then the number of d -simplexes in $\mathcal{S}(\beta_s^d)$ that are adjacent to $\alpha^d(a_1, a_2, \dots, a_{d-1})$ is i .

For $\alpha^d \in \mathcal{S}(\beta_s^d)$ let $A(\alpha^d)$ be the set of d -simplexes in $\mathcal{S}(\beta_s^d)$ that are adjacent to α^d and

$$C_{\alpha^d}(n) = S_{\alpha^d}(n) \setminus \bigcup_{\alpha_1^d \in A(\alpha^d)} S_{\alpha_1^d}(n).$$

Simply speaking, $C_{\alpha^d}(n)$ is formed by successive facet-cuts on $S_{\alpha^d}(n)$. By the geometric description of polytope numbers

$$S_{\beta_s^d}(n) = \biguplus_{\alpha^d \in \mathcal{S}(\beta_s^d)} C_{\alpha^d}(n),$$

and if $\alpha^d \in \mathcal{S}_i$ then successive i facet-cuts on S_{α^d} yield

$$|C_{\alpha^d}(n)| = \alpha^d(n - i).$$

Therefore

$$\begin{aligned} \beta^d(n) &= \beta_s^d(n) = |S_{\beta_s^d}(n)| \\ &= \left| \biguplus_{\alpha^d \in \mathcal{S}(\beta_s^d)} C_{\alpha^d}(n) \right| = \left| \biguplus_{i \in [d-1]_0} \biguplus_{\alpha^d \in \mathcal{S}_i(\beta_s^d)} C_{\alpha^d}(n) \right| \\ &= \sum_{i \in [d-1]_0} \sum_{\alpha^d \in \mathcal{S}_i(\beta_s^d)} |C_{\alpha^d}(n)| = \sum_{i \in [d-1]_0} \sum_{\alpha^d \in \mathcal{S}_i(\beta_s^d)} \alpha^d(n - i) \\ &= \sum_{i \in [d-1]_0} \binom{d-1}{i} \alpha^d(n - i). \end{aligned}$$

This computation of $\beta^d(n)$ coincides with Kim's computation of $\beta^d(n)$ [3].

5.1.2. Measure polytope γ^d . Let S_d be the set of permutations on the set $[d]$. We denote by $[a_1, a_2, \dots, a_n]$ a permutation on $[d]$. For the permutation $[a_1, a_2, \dots, a_d]$ we define $\alpha^d[a_1, a_2, \dots, a_d]$ to be the simplex with the vertex set

$$\{\mathbf{0}, \mathbf{e}_{a_1}, \mathbf{e}_{a_1} + \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_1} + \mathbf{e}_{a_2} + \dots + \mathbf{e}_{a_d} = \mathbf{1}\}.$$

If we define

$$\mathcal{S}(\gamma_s^d) = \{\alpha^d[a_1, a_2, \dots, a_d] \mid [a_1, a_2, \dots, a_d] \in S_d\},$$

then, by a simple reasoning,

$$\gamma_s^d = \bigcup_{\alpha^d \in \mathcal{S}(\gamma_s^d)} \alpha^d.$$

Stanley gave this representation of γ_s^d [4]. We can easily show that the d -simplexes in $\mathcal{S}(\gamma_s^d)$ form a pointed triangulation $\mathcal{C}_{\gamma_s^d}$ such that

$$\begin{cases} |\mathcal{C}_{\gamma_s^d}| = \gamma_s^d \\ \mathcal{C}_{\gamma_s^d} = \bigoplus_{\alpha^d \in \mathcal{S}(\gamma_s^d)} \alpha^d \end{cases}.$$

We call $\mathcal{C}_{\gamma_s^d}$ the *standard pointed triangulation* of γ_s^d and assume that $\gamma^d(n)$ are formed by this pointed triangulation

For $i \in [d-1]_0$ let $\mathcal{S}_i(\gamma_s^d)$ be the set of $\alpha^d[a_1, a_2, \dots, a_d]$ in $\mathcal{S}(\gamma_s^d)$ such that for $j \in [d-1]$ the number of j obeying $a_j > a_{j+1}$ is i . Each of the sets $\mathcal{S}_i(\gamma_s^d)$ satisfies

$$\begin{cases} |\mathcal{S}_i(\gamma_s^d)| = \left\langle \begin{smallmatrix} d \\ i \end{smallmatrix} \right\rangle \\ \mathcal{S}(\gamma_s^d) = \bigoplus_{i \in [d-1]_0} \mathcal{S}_i(\gamma_s^d) \end{cases}$$

where $\left\langle \begin{smallmatrix} d \\ i \end{smallmatrix} \right\rangle$ is the Eulerian number. We say that an $\alpha^d[a_1^1, a_2^1, \dots, a_d^1]$ is *adjacent* to another $\alpha^d[a_1^2, a_2^2, \dots, a_d^2]$ if

$$[a_1^2, a_2^2, \dots, a_d^2] = [a_1^1, \dots, a_{j-1}^1, a_{j+1}^1, a_j^1, a_{j+2}^1, \dots, a_d^1]$$

where $a_j^1 < a_{j+1}^1$. By the definition of adjacency

$$\dim(\alpha^d[a_1^1, a_2^1, \dots, a_d^1] \cap \alpha^d[a_1^2, a_2^2, \dots, a_d^2]) = d-1,$$

and if $\alpha^d[a_1^1, a_2^1, \dots, a_d^1] \in \mathcal{S}_i(\gamma_s^d)$ then $\alpha^d[a_1^2, a_2^2, \dots, a_d^2] \in \mathcal{S}_{i+1}(\gamma_s^d)$. In addition, if $\alpha^d[a_1, a_2, \dots, a_d] \in \mathcal{S}_i(\gamma_s^d)$ then the number of d -simplexes in $\mathcal{S}(\gamma_s^d)$ that are adjacent to $\alpha^d[a_1, a_2, \dots, a_d]$ is i .

For $\alpha^d \in \mathcal{S}(\gamma_s^d)$ let $\mathcal{A}(\alpha^d)$ be the set of d -simplexes in $\mathcal{S}(\gamma_s^d)$ that are adjacent to α^d and

$$C_{\alpha^d}(n) = S_{\alpha^d}(n) \setminus \bigcup_{\alpha_1^d \in \mathcal{A}(\alpha^d)} S_{\alpha_1^d}(n).$$

Simply speaking, $C_{\alpha^d}(n)$ is formed by successive facet-cuts on $S_{\alpha^d}(n)$. By the geometric description of polytope numbers

$$S_{\gamma_s^d}(n) = \bigsqcup_{\alpha^d \in \mathcal{S}(\gamma_s^d)} C_{\alpha^d}(n),$$

and if $\alpha^d \in \mathcal{S}_i(\gamma_s^d)$ then successive i facet-cuts on $S_{\alpha^d}(n)$ yield

$$|C_{\alpha^d}(n)| = \alpha^d(n - i).$$

Therefore

$$\begin{aligned} (5.1) \quad \gamma_s^d(n) &= \gamma_s^d = |S_{\gamma_s^d}(n)| \\ &= \left| \bigsqcup_{\alpha^d \in \mathcal{S}(\gamma_s^d)} C_{\alpha^d}(n) \right| = \left| \bigsqcup_{i \in [d-1]_0} \bigsqcup_{\alpha^d \in \mathcal{S}_i(\gamma_s^d)} C_{\alpha^d}(n) \right| \\ &= \sum_{i \in [d-1]_0} \sum_{\alpha^d \in \mathcal{S}_i(\gamma_s^d)} |C_{\alpha^d}(n)| = \sum_{i \in [d-1]_0} \sum_{\alpha^d \in \mathcal{S}_i(\gamma_s^d)} \alpha^d(n - i) \\ &= \sum_{i \in [d-1]} \left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle \alpha^d(n - i). \end{aligned}$$

This computation of $\gamma^d(n)$ coincides with Kim's computation [3] and we can also obtain this decomposition of $\gamma^d(n)$ by a Worpitzky's result [5].

5.1.3. The product of simplexes. For $j \in [l]$ let $\mathbf{0}^j$ be the zero vector and $\mathbf{e}_0^j, \mathbf{e}_1^j, \dots, \mathbf{e}_{d_j}^j$ be the unit vectors in \mathbb{R}^{d_j+1} . Writing

$$\alpha^{d_j} = \text{conv}\left(\{\mathbf{e}_0^j, \mathbf{e}_1^j, \dots, \mathbf{e}_{d_j}^j\}\right)$$

for $j \in [l]$, we define

$$\alpha^{d_1, d_2, \dots, d_l} = \prod_{j \in [l]} \alpha^{d_j}$$

to be the *standard product of simplexes*. Let $d = \sum_{j \in [l]} d_j$ and S_{d_1, d_2, \dots, d_l} be the set of permutations on the multiset $\{1^{d_1}, 2^{d_2}, \dots, l^{d_l}\}$ where the number of i in this multiset is d_i . For a permutation $[a_1, a_2, \dots, a_d] \in S_{d_1, d_2, \dots, d_l}$ denoting

$$\begin{cases} (A_0^1, A_0^2, \dots, A_0^l) = (0, 0, \dots, 0) \\ (A_i^1, A_i^2, \dots, A_i^l) = \sum_{k \in [i]} \mathbf{e}_{a_k} \end{cases}$$

for $i \in [d]$, we define $\alpha^d[a_1, a_2, \dots, a_d]$ to be the simplex with the vertex set

$$\text{vert}(\alpha^d[a_1, a_2, \dots, a_d]) = \left\{ \prod_{j \in [l]} \mathbf{e}_{A_i^j}^j \mid i \in [d]_0 \right\}$$

and

$$\mathcal{S}(\alpha^{d_1, d_2, \dots, d_l}) = \{ \alpha^{d_1, d_2, \dots, d_l}[a_1, a_2, \dots, a_d] \mid [a_1, a_2, \dots, a_d] \in S_{d_1, d_2, \dots, d_l} \}.$$

We claim that

$$(5.2) \quad \alpha^{d_1, d_2, \dots, d_l} = \bigcup_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} \alpha^d.$$

By definition

$$\text{vert}(\alpha^{d_1, d_2, \dots, d_l}) \supseteq \bigcup_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} \text{vert}(\alpha^d),$$

thus

$$\alpha^{d_1, d_2, \dots, d_l} \supseteq \bigcup_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} \alpha^d.$$

Consider the opposite inclusion \subseteq . Let

$$\mathbf{x} = \prod_{j \in [l]} (x_0^j, x_1^j, \dots, x_{d_j}^j)^* \in \alpha^{d_1, d_2, \dots, d_l}.$$

Denoting $X_{k_j}^j = \sum_{i \in [k_j]_0} x_i^j$ for $(j, k_j) \in [l] \times [d_j]$, we let \prec be the total order on $X_{k_j}^j$ defined by $X_{k_{j_1}}^{j_1} \prec X_{k_{j_2}}^{j_2}$ if one of the following is true:

$$\begin{cases} X_{k_{j_1}}^{j_1} < X_{k_{j_2}}^{j_2} \\ X_{k_{j_1}}^{j_1} = X_{k_{j_2}}^{j_2} \text{ for } j_1 < j_2 \\ X_{k_{j_1}}^{j_1} = X_{k_{j_2}}^{j_2} \text{ for } j_1 = j_2, k_{j_1} < k_{j_2} \end{cases}.$$

Assuming that $X_{k_{j_1}}^{j_1} \prec X_{k_{j_2}}^{j_2} \prec \dots \prec X_{k_{j_d}}^{j_d}$, for $i \in [d]$ we define the l -tuples

$$(A_i^1, A_i^2, \dots, A_i^l) = \sum_{k \in [i]} \mathbf{e}_{j_k}.$$

If we denote

$$\begin{cases} X_0^0 = \sum_{j \in [l]} x_0^j \\ X_{k_{j_0}}^{j_0} = 0 \end{cases},$$

then

$$\mathbf{x} = \sum_{i \in [d]} \left(X_{k_{j_i}}^{j_i} - X_{k_{j_i-1}}^{j_i-1} \right) \prod_{j \in [l]} \mathbf{e}_{A_i^j}^j + X_0^0 \prod_{j \in [l]} \mathbf{e}_0^j.$$

The equation

$$\sum_{i \in [d+1]} \left(X_{k_{j_i}}^{j_i} - X_{k_{j_i-1}}^{j_i-1} \right) + X_0^0 = \sum_{j \in [l]} \sum_{i \in [d_j]_0} x_i^j = 1$$

provides $\mathbf{x} \in \alpha^d[j_1, j_2, \dots, j_d]$, thus

$$\prod_{j \in [l]} \alpha^{d_j} \subseteq \bigcup_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} \alpha^d.$$

From the identity (5.2), we define a polytopal complex

$$\mathcal{C}_{d_1, d_2, \dots, d_l} = \bigoplus_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} \alpha^d.$$

Then, by a simple reasoning, $\mathcal{C}_{d_1, d_2, \dots, d_l}$ is a pointed triangulation of $\prod_{j \in [l]} \alpha^{d_j}$.

We call $\mathcal{C}_{d_1, d_2, \dots, d_l}$ the *standard pointed triangulation* of $\prod_{j \in [l]} \alpha^{d_j}$.

For $i \in [d-1]_0$ let $\mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})$ be the set of $\alpha^d[a_1, a_2, \dots, a_d]$ in $\mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})$ such that for $k \in [d-1]_0$ the number of k satisfying $a_k > a_{k+1}$ is i . We define

$$\left\langle d_1, d_2, \dots, d_l \right\rangle_i = \left| \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l}) \right|.$$

The sets $\mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})$ satisfy

$$\mathcal{S}(\alpha^{d_1, d_2, \dots, d_l}) = \bigsqcup_{i \in [d-1]} \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l}).$$

We say that an $\alpha^d[a_1^1, a_2^1, \dots, a_d^1]$ is *adjacent* to another $\alpha^d[a_1^2, a_2^2, \dots, a_d^2]$ if

$$[a_1^2, a_2^2, \dots, a_d^2] = [a_1^1, \dots, a_{k-1}^1, a_{k+1}^1, a_k^1, a_{k+2}^1, \dots, a_d^1]$$

where $a_k^1 < a_{k+1}^1$. By the definition of adjacency

$$\dim \left(\alpha^d[a_1^1, a_2^1, \dots, a_d^1] \cap \alpha^d[a_1^2, a_2^2, \dots, a_d^2] \right) = d-1,$$

and if $\alpha^d[a_1^1, a_2^1, \dots, a_d^1] \in \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})$ then

$$\alpha^d[a_1^2, a_2^2, \dots, a_d^2] \in \mathcal{S}_{i+1}(\alpha^{d_1, d_2, \dots, d_l}).$$

In addition, if $\alpha^d[a_1, a_2, \dots, a_d] \in \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})$ then the number of d -simplexes in $\mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})$ that are adjacent to $\alpha^d[a_1, a_2, \dots, a_d]$ is i .

For $\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})$ let $\mathcal{A}(\alpha^d)$ be the set of d -simplexes in $\mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})$ that are adjacent to α^d and let

$$C_{\alpha^d}(n) = S_{\alpha^d}(n) \setminus \bigcup_{\alpha_1^d \in \mathcal{A}(\alpha^d)} S_{\alpha_1^d}(n).$$

Simply speaking, $C_{\alpha^d}(n)$ is formed by successive facet-cuts on $S_{\alpha^d}(n)$. By the geometric description of polytope numbers

$$S_{\alpha^{d_1, d_2, \dots, d_l}}(n) = \biguplus_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} C_{\alpha^d}(n),$$

and if $\alpha^d \in \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})$ then successive i facet-cuts on S_{α^d} yields

$$|C_{\alpha^d}(n)| = \alpha^d(n - i).$$

Therefore

(5.3)

$$\begin{aligned} \alpha^{d_1, d_2, \dots, d_l}(n) &= |S_{\alpha^{d_1, d_2, \dots, d_l}}(n)| \\ &= \left| \biguplus_{\alpha^d \in \mathcal{S}(\alpha^{d_1, d_2, \dots, d_l})} C_{\alpha^d}(n) \right| = \left| \biguplus_{i \in [d-1]} \biguplus_{\alpha^d \in \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})} C_{\alpha^d}(n) \right| \\ &= \sum_{i \in [d-1]} \sum_{\alpha^d \in \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})} |C_{\alpha^d}(n)| = \sum_{i \in [d-1]} \sum_{\alpha^d \in \mathcal{S}_i(\alpha^{d_1, d_2, \dots, d_l})} \alpha^d(n - i) \\ &= \sum_{i \in [d-1]} \left\langle d_1, d_2, \dots, d_l \right\rangle_i \alpha^d(n - i). \end{aligned}$$

Remark 5.1. If $d_1 = d_2 = \dots = d_l = 1$, then $\alpha^{d_1, d_2, \dots, d_l}$ is an l -measure polytope. Thus the identity (5.3) is a generalization of the identity (5.1). In this point of view, we call the numbers $\left\langle d_1, d_2, \dots, d_l \right\rangle_i$ generalized Eulerian numbers.

5.2. Decomposition theorem 2 and Decomposition theorem 3. Essentially, the same phenomenon describes Decomposition theorems 2 and 3. Therefore we apply these theorems at the same time.

5.2.1. Simplex α^d . For α_s^d let $\mathbf{v}_{\alpha^d} = \mathbf{e}_0$. Suppose that $\mathcal{C}_{\alpha_s^d}$ is the $V(\alpha_s^d)$ -pointed triangulation. Since a k -simplex in $\mathcal{C}_{\alpha_s^d}$ that contains \mathbf{v}_{α^d} is determined by k vertexes among $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$, the number of such k -faces in $\mathcal{C}_{\alpha_s^d}$

is $\binom{d}{k}$. Therefore, by Decomposition theorem 3-1,

$$\alpha^d(n) = \alpha_s^d(n) = \sum_{k \in [d]_0} \binom{d}{k} \alpha^k(n-k).$$

5.2.2. *Cross polytope* β^d . For β_s^d let $\mathbf{v}_{\beta^d} = \mathbf{e}_d$. An r -simplex α^r in $\mathcal{C}_{\beta_s^d}$ satisfies $\alpha^r \cap \text{int}(\beta^d) \neq \emptyset$ if and only if $\{\mathbf{e}_d, -\mathbf{e}_d\} \subseteq \alpha^r$, thus

$$\alpha^r = \{\mathbf{e}_d, -\mathbf{e}_d, a_{i_1}\mathbf{e}_{i_1}, a_{i_2}\mathbf{e}_{i_2}, \dots, a_{i_{r-1}}\mathbf{e}_{i_{r-1}}\}$$

where $a_{i_j} \in \{1, -1\}$. By the way, the number of such r -simplexes in \mathcal{C}_{β^d} is $\binom{d-1}{r-1}2^{r-1}$, thus Decomposition theorem 2 yields

$$\beta^d(n) = \beta_s^d(n) = \sum_{r \in [d]} (-1)^{d-r} \binom{d-1}{r-1} 2^{r-1} \alpha^r(n).$$

This computation of $\beta^d(n)$ coincides with Kim's computation of $\beta^d(n)$ [3]. Similarly, Decomposition theorem 3-2 provides

$$\beta^d(n) = \sum_{k \in [d]} \binom{d-1}{k-1} 2^{k-1} \alpha^k(n-k+1).$$

In addition, if we consider the number of k -simplexes in $\mathcal{C}_{\beta_s^d}$ that are on the boundary of β_s^d and contain $\mathbf{v}_{\beta_s^d}$, which is $2^k \binom{d-1}{k}$ by a simple computation, then, by Decomposition theorem 3-1,

$$\beta^d(n) = \sum_{k \in [d]_0} \left\{ 2^k \binom{d-1}{k} + 2^{k-1} \binom{d-1}{k-1} \right\} \alpha^k(n-k).$$

5.2.3. *Measure polytope* γ^d . For γ_s^d let $\mathbf{v}_{\gamma^d} = \mathbf{0}$. If α^r is an r -simplex in \mathcal{C}_{γ^d} satisfying $\alpha^r \cap \text{int}(\gamma^d) \neq \emptyset$, then $\{\mathbf{0}, \mathbf{1}\} \subseteq \alpha^r$. Moreover, a d -simplex $\alpha^d[a_1, a_2, \dots, a_d] \in \mathcal{C}_{\gamma^d}$ satisfying $\alpha^r \subseteq \alpha^d[a_1, a_2, \dots, a_d]$ exists, thus the vertexes of α^r are

$$\mathbf{0}, \sum_{k_1 \in [d_1]} \mathbf{e}_{a_{k_1}}, \sum_{k_2 \in [d_2]} \mathbf{e}_{a_{k_2}}, \dots, \sum_{k_{r-1} \in [d_{r-1}]} \mathbf{e}_{a_{k_{r-1}}}, \mathbf{1}$$

where $1 \leq d_1 < d_2 < \dots < d_{r-1} \leq d-1$. Consequently, the vertexes of α^r correspond to an ordered partition of $[d]$ into r sets. The number of such partitions is $r!S(d, r)$ where $S(d, r)$ is the Stirling number of the second kind, which is the number of ways to partition a d -set into r sets. If we assume that

$$a_1 > a_2 > \dots > a_{d_1}, a_{d_1+1} > a_{d_1+2} > \dots > a_{d_2}, \dots, a_{d_{r-1}+1} > a_{d_{r-1}+2} > \dots > a_d,$$

that is, $[a_1, a_2, \dots, a_d]$ has at least $d - r$ descents, then the number of such α^r is

$$\sum_{i \in [d-1] \setminus [d-1-r]} \left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle \binom{i}{d-r}.$$

Therefore Decomposition theorem 2 yields

$$\begin{aligned} \gamma^d(n) &= \gamma_s^d(n) = \sum_{r \in [d]} (-1)^{d-r} r! S(d, r) \alpha^r(n) \\ &= \sum_{r \in [d]} (-1)^{d-r} \left\{ \sum_{i \in [d-1] \setminus [d-1-r]} \left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle \binom{i}{d-r} \right\} \alpha^r(n). \end{aligned}$$

Similarly, by Decomposition theorem 3-2,

$$\begin{aligned} \gamma^d(n) &= \sum_{r \in [d]} r! S(d, r) \alpha^r(n - r + 1) \\ &= \sum_{r \in [d]} \left\{ \sum_{i \in [d-1] \setminus [d-1-r]} \left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle \binom{i}{d-r} \right\} \alpha^r(n - r + 1). \end{aligned}$$

Let α^r be an r -simplex in $\mathcal{C}_{\gamma_s^d}$ that contains $\mathbf{0}$ and is a subpolytope of $\alpha^d[a_1, a_2, \dots, a_d]$. Then

$$\alpha^r = \text{vert} \left(\left\{ \mathbf{0}, \sum_{k_1 \in [d_1]} \mathbf{e}_{a_{k_1}}, \sum_{k_2 \in [d_2]} \mathbf{e}_{a_{k_2}}, \dots, \sum_{k_{r-1} \in [d_{r-1}]} \mathbf{e}_{a_{k_{r-1}}} \right\} \right)$$

where $1 \leq d_1 < d_2 < \dots < d_{r-1} \leq d$. Thus to choose an r -simplex in $\mathcal{C}_{\gamma_s^d}$ is equivalent to construct an ordered partition of a subset of $[d]$ with j elements into r sets where $j \in [d] \setminus [r-1]$. It follows that the number of such r -simplexes in $\mathcal{C}_{\gamma_s^d}$ is

$$\sum_{j \in [d] \setminus [r-1]} \binom{d}{j} r! S(j, r).$$

Decomposition theorem 3-1 yields

$$\gamma^d(n) = \sum_{r \in [d]_0} \left(\sum_{j \in [d] \setminus [r-1]} \binom{d}{j} r! S(j, r) \right) \alpha^r(n - r).$$

5.2.4. *The product of simplexes.* For $\alpha^{d_1, d_2, \dots, d_l}$ let

$$\mathbf{v}_{\alpha^{d_1, d_2, \dots, d_l}} = \prod_{j \in [l]} \mathbf{e}_0^j.$$

Suppose that $\alpha^r \subseteq \alpha^d[a_1, a_2, \dots, a_d]$ and for $i_1 < i_2 < \dots < i_r$ let

$$\prod_{j \in [l]} \mathbf{e}_0^j, \prod_{j \in [l]} \mathbf{e}_{A_{i_1}^j}^j, \prod_{j \in [l]} \mathbf{e}_{A_{i_2}^j}^j, \dots, \prod_{j \in [l]} \mathbf{e}_{A_{i_r}^j}^j$$

be the vertexes of α^r . The simplex α^r satisfies $\alpha^r \cap \text{int}(\alpha^{d_1, d_2, \dots, d_l}) \neq \emptyset$ if and only if there is a point $\mathbf{x} = \prod_{j \in [l]} (x_0^j, x_1^j, \dots, x_{d_j}^j)^*$ in α^r such that every entry of \mathbf{x} is nonzero, equivalently,

$$(5.4) \quad \{0, A_{i_1}^j, A_{i_2}^j, \dots, A_{i_r}^j\} = [d_j]_0$$

for $j \in [l]$. Therefore, if we denote $(A_{i_0}^1, A_{i_0}^2, \dots, A_{i_0}^l) = \mathbf{0}$, then the number of r -simplexes α^r satisfying $\alpha^r \cap \text{int}(\alpha^{d_1, d_2, \dots, d_l}) \neq \emptyset$ is the number of elements in the set \mathcal{A} composed of

$$\left((A_{i_1}^1, A_{i_1}^2, \dots, A_{i_1}^l), (A_{i_2}^1, A_{i_2}^2, \dots, A_{i_2}^l), \dots, (A_{i_r}^1, A_{i_r}^2, \dots, A_{i_r}^l) \right)$$

that satisfies

$$(5.5) \quad \begin{cases} (A_{i_m}^1, A_{i_m}^2, \dots, A_{i_m}^l) - (A_{i_{m-1}}^1, A_{i_{m-1}}^2, \dots, A_{i_{m-1}}^l) > \mathbf{0} & \text{for } m \in [r] \\ \{0, A_{i_1}^j, A_{i_2}^j, \dots, A_{i_r}^j\} = [d_j]_0 & \text{for } j \in [l] \end{cases}.$$

For $k \in [r]$ let \mathcal{A}_k be the subset of \mathcal{A} each of whose elements satisfies

$$(A_{i_k}^1, A_{i_k}^2, \dots, A_{i_k}^l) - (A_{i_{k-1}}^1, A_{i_{k-1}}^2, \dots, A_{i_{k-1}}^l) = \mathbf{0}.$$

The number of

$$\left((A_{i_1}^1, A_{i_1}^2, \dots, A_{i_1}^l), (A_{i_2}^1, A_{i_2}^2, \dots, A_{i_2}^l), \dots, (A_{i_r}^1, A_{i_r}^2, \dots, A_{i_r}^l) \right)$$

satisfying the condition (5.5) is the number of elements in $\mathcal{A} \setminus \left(\bigcup_{k \in [r]} \mathcal{A}_k \right)$. By the principle of inclusion and exclusion [4]

$$\left| \mathcal{A} \setminus \left(\bigcup_{k \in [r]} \mathcal{A}_k \right) \right| = \sum_{k \in [r]_0} (-1)^k \binom{r}{k} \prod_{j \in [l]} \binom{r-k}{d_j},$$

thus Decomposition theorems 2 and 3-2 yield

$$\begin{cases} \alpha^{d_1, d_2, \dots, d_l}(n) = \sum_{r \in [d]} (-1)^{d-r} \left\{ \sum_{k \in [r]_0} (-1)^k \binom{r}{k} \prod_{j \in [l]} \binom{r-k}{d_j} \right\} \alpha^r(n) \\ \alpha^{d_1, d_2, \dots, d_l}(n) = \sum_{r \in [d]} \left\{ \sum_{k \in [r]_0} (-1)^k \binom{r}{k} \prod_{j \in [l]} \binom{r-k}{d_j} \right\} \alpha^r(n-r+1) \end{cases},$$

respectively.

We can similarly compute the number of r -simplexes in $\mathcal{C}_{d_1, d_2, \dots, d_l}$ that contain $\mathbf{v}_{\alpha^{d_1, d_2, \dots, d_l}}$. It is the number of

$$\left((A_{i_1}^1, A_{i_1}^2, \dots, A_{i_1}^l), (A_{i_2}^1, A_{i_2}^2, \dots, A_{i_2}^l), \dots, (A_{i_r}^1, A_{i_r}^2, \dots, A_{i_r}^l) \right)$$

obeying

$$\begin{cases} (A_{i_m}^1, A_{i_m}^2, \dots, A_{i_m}^l) - (A_{i_{m-1}}^1, A_{i_{m-1}}^2, \dots, A_{i_{m-1}}^l) > \mathbf{0} & \text{for } m \in [r] \\ \{0, A_{i_1}^j, A_{i_2}^j, \dots, A_{i_r}^j\} = [d_j^1]_0 & \text{for } j \in [l] \text{ and } d_j^1 \in [d_j]_0 \end{cases}.$$

By the principle of inclusion and exclusion, this number is

$$\sum_{k \in [r]_0} (-1)^k \binom{r}{k} \prod_{j \in [l]} \binom{d_j + r - k}{r - k},$$

therefore Decomposition theorem 3-1 furnishes

$$\alpha^{d_1, d_2, \dots, d_l}(n) = \sum_{r \in [d]_0} \left\{ \sum_{k \in [r]_0} (-1)^k \binom{r}{k} \prod_{j \in [l]} \binom{d_j + r - k}{r - k} \right\} \alpha^r(n - r).$$

6. APPLICATIONS OF POLYTOPES NUMBERS

Decomposition theorems are methods to represent polytope numbers by sums of simplex numbers and the vertex description of polytope numbers gives a relation between polytope numbers and a set of chains in posets formed by faces of polytopes. Using these facts, we consider applications of polytope numbers to several research topics. These topics are composed of generalized Eulerian numbers, lattice paths, plane partitions, and Young tableaux. For two polytopes P_1 and P_2 we say that P_1 is a vertex subpolytope of P_2 if $\text{vert}(P_1) \subseteq \text{vert}(P_2)$.

6.1. Generalized Eulerian numbers. In this subsection, we denote the unit vectors of \mathbb{R}^{d+1} by $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d$ and we define

$$\mathbf{e}_{i_1, i_2, \dots, i_l} = \mathbf{e}_{i_1} \times \mathbf{e}_{i_2} \times \dots \times \mathbf{e}_{i_l}$$

where $(i_1, i_2, \dots, i_l) \in [d_1]_0 \times [d_2]_0 \times \dots \times [d_l]_0$.

Letting $d = \sum_{i \in [l]} d_i$, we define

$$L(i_1, i_2, \dots, i_{d+1}) = \{\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_{d+1}}\}$$

to be the lattice path

$$\mathbf{0} \rightarrow \mathbf{e}_{i_1} \rightarrow \mathbf{e}_{i_1} + \mathbf{e}_{i_2} \rightarrow \dots \rightarrow \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \dots + \mathbf{e}_{i_d}$$

from $(0, 0, \dots, 0)$ to (d_1, d_2, \dots, d_l) , and $\mathcal{L}(d_1, d_2, \dots, d_l)$ to be the set of such $L(i_1, i_2, \dots, i_{d+1})$. For $L(i_1, i_2, \dots, i_{d+1}) \in \mathcal{L}(d_1, d_2, \dots, d_l)$ let

$$\alpha^d(i_1, i_2, \dots, i_{d+1}) = \text{conv}\left(\{\mathbf{e}_{j_1, j_2, \dots, j_l} \mid (j_1, j_2, \dots, j_l) \in L(i_1, i_2, \dots, i_{d+1})\}\right).$$

be a d -simplex, which is a vertex subpolytope of $\alpha^{d_1, d_2, \dots, d_l}$.

For a lattice path $L(i_1, i_2, \dots, i_{d+1})$ in $\mathcal{L}(d_1, d_2, \dots, d_l)$, we define a descent of $L(i_1, i_2, \dots, i_{d+1})$ to be an index p such that $i_p > i_{p+1}$. Then we can easily show that

$$(6.1) \quad \prod_{i \in [l]} \alpha^{d_i} = \bigcup_{L(i_1, i_2, \dots, i_{d+1}) \in \mathcal{L}(d_1, d_2, \dots, d_l)} \alpha^d(i_1, i_2, \dots, i_{d+1}).$$

By Decomposition theorem 1,

$$\prod_{i \in [l]} \alpha^{d_i}(n) = \sum_{i \in [d-1]_0} a_i \alpha^d(n-i)$$

where a_i is the number of lattice paths in $\mathcal{L}(d_1, d_2, \dots, d_l)$ with i descents. Since $a_i = \left\langle \begin{smallmatrix} l \\ i \end{smallmatrix} \right\rangle$ for $d_1 = d_2 = \dots = d_l = 2$, the numbers a_0, a_1, \dots, a_{d-1} are *generalized Eulerian numbers*. By

$$\begin{pmatrix} \prod_{i \in [l]} \alpha^{d_i}(1) \\ \prod_{i \in [l]} \alpha^{d_i}(2) \\ \dots \\ \prod_{i \in [l]} \alpha^{d_i}(d) \end{pmatrix} = \begin{pmatrix} \alpha^d(1) & 0 & \dots & 0 \\ \alpha^d(2) & \alpha^d(1) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha^d(d) & \alpha^d(d-1) & \dots & \alpha^d(1) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{d-1} \end{pmatrix},$$

the numbers a_0, a_1, \dots, a_{d-1} are

$$\begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{d-1} \end{pmatrix} = \begin{pmatrix} (-1)^0 \binom{d+1}{0} & 0 & \dots & 0 \\ (-1)^1 \binom{d+1}{1} & (-1)^0 \binom{d+1}{0} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (-1)^{d-1} \binom{d+1}{d-1} & (-1)^{d-2} \binom{d+1}{d-2} & \dots & (-1)^0 \binom{d+1}{0} \end{pmatrix} \begin{pmatrix} \prod_{i \in [l]} \alpha^{d_i}(1) \\ \prod_{i \in [l]} \alpha^{d_i}(2) \\ \dots \\ \prod_{i \in [l]} \alpha^{d_i}(d) \end{pmatrix}.$$

Therefore

$$a_i = \sum_{j \in [i]} (-1)^j \binom{d+1}{j} \prod_{k \in [l]} \alpha^{d_k}(i+1-j).$$

6.2. Lattice paths and plane partitions. Let P_L be a d -dimensional vertex subpolytope of $\alpha^{d_1, d_2, \dots, d_l}$ with $d_1 \geq d_2 \geq \dots \geq d_l$ each of whose vertexes $\mathbf{e}_{i_1, i_2, \dots, i_l}$ satisfies $i_1 \geq i_2 \geq \dots \geq i_l$. If we use the decomposition (6.1), then the d -simplexes $\alpha^d(i_1, i_2, \dots, i_{d+l})$ in the decomposition (6.1) correspond to the lattice paths from $(0, 0, \dots, 0)$ to (d_1, d_2, \dots, d_l) such that every point (x_1, x_2, \dots, x_d) in these lattice paths satisfies $x_1 \geq x_2 \geq \dots \geq x_l$. By Decomposition theorem 1,

$$P_L(n) = \sum_{i \in [d-1]} a_i \alpha^d(n-i)$$

where a_i is the number of simplexes $\alpha^d(i_1, i_2, \dots, i_{d+1})$ such that the number of descents in $L(i_1, i_2, \dots, i_{d+1})$ is i and every point (x_1, x_2, \dots, x_l) in the lattice path $L(i_1, i_2, \dots, i_{d+1})$ satisfies $x_1 \geq x_2 \geq \dots \geq x_l$. Krattenthaler computed the number $\sum_{i \in [d-1]_0} a_i$ [6] and a_i are refinements of $\sum_{i \in [d-1]_0} a_i$.

We now compute the numbers a_i . According to the decomposition (6.1), we can define a partial order of the vertexes in P_L by

$$\mathbf{e}_{a_{11}, a_{12}, \dots, a_{1l}} \geq \mathbf{e}_{a_{21}, a_{22}, \dots, a_{2l}}$$

if

$$a_{11} \leq a_{21}, a_{12} \leq a_{22}, \dots, a_{1l} \leq a_{2l}.$$

Therefore we need to consider the number of different sums of $n-1$ vertexes $\mathbf{e}_{a_{i1}, a_{i2}, \dots, a_{il}}$ for $i \in [n-1]$ in P_L such that

$$\mathbf{e}_{a_{i1}, a_{i2}, \dots, a_{il}} \geq \mathbf{e}_{a_{i+1,1}, a_{i+1,2}, \dots, a_{i+1,l}}$$

to use the vertex description of polytope numbers. By a simple computation, $P_L(n)$ is the number of plane partitions with entries a_{ij} satisfying $a_{ij} \in [d_1]$. Since the number of such plane partitions is

$$P(n-1, l, d_1) = \prod_{i \in [n-1]} \prod_{j \in [l]} \prod_{k \in [d_1]} \frac{i+j+k-1}{i+j+k-2},$$

where $P(0, l, d_1) = 1$ [7], the vertex description of polytope numbers for P_L yields

$$P_L(n) = P(n-1, 1, d_1).$$

The method to compute coefficients in the decomposition form of Decomposition theorem 1 yields

$$\begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{d-1} \end{pmatrix} = \begin{pmatrix} (-1)^0 \binom{d+1}{0} & 0 & \dots & 0 \\ (-1)^1 \binom{d+1}{1} & (-1)^0 \binom{d+1}{0} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (-1)^{d-1} \binom{d+1}{d-1} & (-1)^{d-2} \binom{d+1}{d-2} & \dots & (-1)^0 \binom{d+1}{0} \end{pmatrix} \begin{pmatrix} P(0, l, d_1) \\ P(1, l, d_1) \\ \dots \\ P(d-1, l, d_1) \end{pmatrix}.$$

Therefore

$$a_i = \sum_{j \in [i]_0} (-1)^j \binom{d+1}{j} P(i-j, l, d_1).$$

If we let $l = 2$, then a_i is the number of lattice paths from $(0, 0)$ to (d_1, d_2) each of whose unit paths is either east or north and the first unit path is east, which never crosses the line $x_1 = x_2$, and whose number of consecutive north-east paths is i . The numbers a_i are refinements of the Lobb number $\sum_{i \in [d-1]_0} a_i$ [15]. In particular, if $d_1 = d_2$ then $a_i = N(d_1, i+1)$ is the Narayana number [7, 8].

If we let $d_1 = d_2 = \dots = d_l$, then a_i are *higher dimensional Narayana numbers* [9].

6.3. Young tableaux and plane partitions. Let P_Y be a vertex subpolytope of $\gamma^{l \cdot m} = \prod_{i \in [l \cdot m]} \text{conv}(\{\mathbf{e}_0, \mathbf{e}_1\})$ whose vertexes are

$$\prod_{(i,j,a_{ij}) \in S} \mathbf{e}_{a_{ij}}$$

where S is a subset of $[l] \times [m] \times [1]_0$ such that each pair of $(i_1, j_1, a_{i_1 j_1})$ and $(i_2, j_2, a_{i_2 j_2})$ in S satisfies that

$$a_{i_1 j_1} \geq a_{i_2 j_2} \text{ if and only if } i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$

Then P_Y is an $(l \cdot m)$ -polytope and there is a one-to-one correspondence between the vertexes of P_Y and the partitions whose size of largest part is at most m and the number of parts is at most l . Note that a partition λ is a finite sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_n$ satisfying

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

and we call $\lambda_1, \lambda_2, \dots, \lambda_n$ the parts of λ and n the number of parts in λ . Using these two properties of the polytope P_Y , we can represent every vertex of P_Y as follows: Let (s_1, s_2, \dots, s_l) be a partition whose parts are s_1, s_2, \dots, s_l with $m \geq s_1 \geq s_2 \geq \dots \geq s_l$. We define

$$\mathbf{v}(s_1, s_2, \dots, s_l) = \prod_{(i,j,a_{ij}) \in S} \mathbf{e}_{a_{ij}}$$

where

$$a_{ij} = \begin{cases} 1 & \text{for } j \in [s_i] \\ 0 & \text{for } j \in [m] \setminus [s_i] \end{cases}$$

when $i \in [l]$. We assign a partial order to $\text{vert}(P_Y)$ by

$$\mathbf{v}(s_1^1, s_2^1, \dots, s_l^1) \geq \mathbf{v}(s_1^2, s_2^2, \dots, s_l^2) \text{ if } s_1^1 \leq s_1^2, s_2^1 \leq s_2^2, \dots, s_l^1 \leq s_l^2.$$

Let $\mathcal{Y}(P_Y)$ be the set of

$$\left\{ \mathbf{v}(s_1^0, s_2^0, \dots, s_l^0), \mathbf{v}(s_1^1, s_2^1, \dots, s_l^1), \dots, \mathbf{v}(s_1^{l \cdot m}, s_2^{l \cdot m}, \dots, s_l^{l \cdot m}) \right\} \subseteq \text{vert}(P_Y)$$

satisfying

$$\begin{cases} (s_1^0, s_2^0, \dots, s_l^0) = (0, 0, \dots, 0) \\ (s_1^i, s_2^i, \dots, s_l^i) - (s_1^{i-1}, s_2^{i-1}, \dots, s_l^{i-1}) = \mathbf{e}_j \text{ for } i \in [l \cdot m] \end{cases}$$

for some $j \in [l]$. For $Y \in \mathcal{Y}(P_Y)$ we define an $(l \cdot m)$ -simplex

$$\alpha_Y^{l \cdot m} = \text{conv}(Y).$$

Since P_Y is a vertex subpolytope of the product of simplexes, decompositions of the product of simplexes gives

$$P_Y = \bigcup_{Y \in \mathcal{Y}(P_Y)} \alpha_Y^{l \cdot m}.$$

Moreover, $\bigoplus_{Y \in \mathcal{Y}(P_Y)} \alpha_Y^{l \cdot m}$ is a pointed triangulation of P_Y , therefore the combination of these results and the vertex description of polytope numbers yields

$$P_Y(n) = P(n-1, l, m).$$

For each $Y \in \mathcal{Y}(P_Y)$ we construct an $l \times m$ Young tableau $T(Y)$ with entries in $[l \cdot m]$ whose entries are strictly decreasing in each row and column

as follows: For an element $\mathbf{v}(s_1, s_2, \dots, s_l)$ of Y we define an $l \times m$ matrix $M(\mathbf{v}(s_1, s_2, \dots, s_l))$ with entries

$$M(\mathbf{v}(s_1, s_2, \dots, s_l))_{ij} = \begin{cases} 1 & \text{for } j \in [s_i] \\ 0 & \text{for } j \in [m] \setminus [s_i] \end{cases}$$

where $i \in [l]$. The the (i, j) entry of $T(Y)$ is the number of $\mathbf{v} \in Y$ such that $M(\mathbf{v})_{ij} = 1$. Then $T(Y)$ is an $l \times m$ Young tableau with entries in $[l \cdot m]$ where each row and column of $T(Y)$ has strictly decreasing entries.

Let

$$Y = \left\{ \mathbf{v}(s_1^0, s_2^0, \dots, s_l^0), \mathbf{v}(s_1^1, s_2^1, \dots, s_l^1), \dots, \mathbf{v}(s_1^{l \cdot m}, s_2^{l \cdot m}, \dots, s_l^{l \cdot m}) \right\}$$

be an element of $\mathcal{Y}(P)$ with

$$\begin{cases} (s_1^0, s_2^0, \dots, s_l^0) = (0, 0, \dots, 0) \\ (s_1^i, s_2^i, \dots, s_l^i) - (s_1^{i-1}, s_2^{i-1}, \dots, s_l^{i-1}) = \mathbf{e}_j \text{ for } i \in [l] \end{cases}$$

for some $j \in [l]$. We define a *descent* of $T(Y)$ to be an index $i \in [l \cdot m - 1]$ such that if

$$\begin{cases} (s_1^{i-1}, s_2^{i-1}, \dots, s_l^{i-1}) - (s_1^i, s_2^i, \dots, s_l^i) = \mathbf{e}_{j_1} \\ (s_1^i, s_2^i, \dots, s_l^i) - (s_1^{i+1}, s_2^{i+1}, \dots, s_l^{i+1}) = \mathbf{e}_{j_2} \end{cases}$$

then $j_1 > j_2$. If we use both the result in decompositions of the product of simplexes and that in Decomposition theorem 1, then the coefficient a_i in

$$P_Y(n) = \sum_{i \in [l \cdot m - 1]_0} a_i \alpha^{l \cdot m} (n - i)$$

is the number of $l \times m$ Young tableaux that have exactly i descents. The method of computing coefficients in Decomposition theorem 1 yields

$$\begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{l \cdot m - 1} \end{pmatrix} = A \begin{pmatrix} P(0, l, m) \\ P(1, l, m) \\ \dots \\ P(l \cdot m - 1, l, m) \end{pmatrix},$$

where

$$A = \begin{pmatrix} (-1)^0 \binom{l \cdot m + 1}{0} & 0 & \cdots & 0 \\ (-1)^1 \binom{l \cdot m + 1}{1} & (-1)^0 \binom{l \cdot m + 1}{0} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ (-1)^{l \cdot m - 1} \binom{l \cdot m + 1}{l \cdot m - 1} & (-1)^{l \cdot m - 2} \binom{l \cdot m + 1}{l \cdot m - 2} & \cdots & (-1)^0 \binom{l \cdot m + 1}{0} \end{pmatrix}.$$

Therefore

$$a_i = \sum_{j \in [i]_0} (-1)^j \binom{l \cdot m + 1}{j} P(i - j, l, m).$$

The number of $l \times m$ Young tableaux with entries in $[l \cdot m]$ is

$$(l \cdot m)! \frac{\prod_{j \in [m-1]} j!}{\prod_{k \in [l+m-1]} k!}$$

by the hook length formula [10], thus

$$\begin{aligned} \sum_{i \in [l \cdot m - 1]_0} a_i &= \sum_{i \in [l \cdot m - 1]_0} \sum_{j \in [i]_0} (-1)^j \binom{l \cdot m + 1}{j} P(i - j, l, m) \\ &= (l \cdot m)! \frac{\prod_{j \in [m-1]} j!}{\prod_{k \in [l+m-1]} k!}. \end{aligned}$$

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